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Structures symplectiques avec décalage adaptées à un feuilletage

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"What [people] really want is usually not some collection of 'answers'—what they want is understanding."

— William Thurston, [Thu94, §1]

RÉSUMÉ

Dans cette thèse, nous travaillons en géométrie algébrique dérivée en caractéristique nulle et étudions les feuilletages dérivés ainsi que les structures symplectiques avec décalage sur de tels objets. Notre premier résultat principal est un théorème d'existence pour les feuilletages dérivés sur les champs de morphismes. Plus précisément, si S est un champ dérivé de base, X est un champ dérivé sur S schématique-propre, plat et localement d'intersection complète, et Y est un champ dérivé de Deligne-Mumford relatif sur S muni d'un feuilletage dérivé, alors le champ dérivé des morphismes $\underline{\mathbf{Map}}_S(X, Y)$ hérite lui-même, sous des hypothèses naturelles de représentabilité, d'un feuilletage dérivé. Afin d'établir ce résultat, nous montrons d'abord que l'∞-catégorie des feuilletages dérivés relatifs peut être réalisée comme une sous-catégorie pleine d'une ∞-catégorie de champs dérivés munis d'une structure supplémentaire, puis nous démontrons que la restriction de Weil le long d'un morphisme schématique-propre, plat et localement d'intersection complète préserve cette structure. On en déduit, plus généralement, un foncteur d'image directe pour les feuilletages dérivés, ainsi qu'une description explicite de leurs complexes tangents après image directe. Comme applications, nous obtenons des feuilletages dérivés sur certains espaces de modules dérivés, notamment les espaces de modules dérivés de courbes stables et les schémas de Hilbert dérivés.

La seconde partie de la thèse introduit les nouvelles notions de structures symplectiques décalées sur les feuilletages dérivés et de feuilletages dérivés lagrangiens. Nous démontrons ensuite des théorèmes d'existence pour ces structures : un théorème d'intersection pour les feuilletages dérivés lagrangiens dans un champ dérivé symplectique n-décalé, produisant un feuilletage dérivé symplectique $(n-1)$-décalé, ainsi qu'un théorème d'image directe symplectique, montrant que l'image directe d'un feuilletage dérivé symplectique n-décalé le long d'un morphisme d-orienté porte naturellement une structure symplectique $(n-d)$-décalée.

ABSTRACT

In this thesis, we work in derived algebraic geometry in characteristic zero and study derived foliations and shifted symplectic structures on such objects. Our first main result is an existence theorem for derived foliations on mapping stacks. More precisely, if S is a base derived stack, X is a proper-schematic, flat, and local complete intersection derived stack over S, and Y is a relative derived Deligne-Mumford stack over S equipped with a derived foliation, then the derived mapping stack $\underline{\mathbf{Map}}_S(X, Y)$ itself inherits, under natural representability hypotheses, a derived foliation. In order to establish this result, we first show that the ∞-category of relative derived foliations can be realized as a full subcategory of an ∞-category of derived stacks equipped with extra structure, and then prove that the Weil restriction along a proper-schematic, flat, and local complete intersection morphism preserves this structure. This yields, more generally, a push-forward functor for derived foliations, together with an explicit description of their tangent complexes after push-forward. As applications, we obtain derived foliations on certain derived moduli spaces, including derived moduli spaces of stable curves and derived Hilbert schemes.

The second part of the thesis introduces the new notions of shifted symplectic structures on derived foliations and Lagrangian derived foliations. We then prove existence theorems for these structures: an intersection theorem for Lagrangian derived foliations in an n-shifted symplectic derived stack, producing an $(n-1)$-shifted symplectic derived foliation, and a symplectic push-forward theorem, showing that the push-forward of an n-shifted symplectic derived foliation along a d-oriented morphism naturally carries an $(n-d)$-shifted symplectic structure.

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CONTENTS

CHAPITRE 1

INTRODUCTION EN FRANÇAIS

Dans cette thèse, nous travaillons en géométrie algébrique dérivée en caractéristique nulle.[1] Nous démontrons des théorèmes d'existence pour les feuilletages dérivés, définissons des structures symplectiques avec décalage sur ceux-ci, et établissons également des théorèmes d'existence pour ces structures. La géométrie algébrique dérivée a été introduite par Bertrand Toën et Gabriele Vezzosi dans [HAGI ; HAGII], et par Jacob Lurie dans [LuriePhD]. Les feuilletages dans ce contexte ont été introduits et étudiés dans une série d'articles de Bertrand Toën et Gabriele Vezzosi, [TV23a] et [TV20], puis dans leur livre [TV25a] ; tandis que les structures symplectiques avec décalage ont été définies et étudiées dans [PTVV] et [CPTVV]. Motivons et introduisons maintenant ces objets.

1.1 UNE MOTIVATION PHYSIQUE DES FEUILLETAGES

En physique, on étudie un système physique en examinant les différents états dans lesquels il peut se trouver. Ces états se distinguent les uns des autres par des quantités mesurables du système : par exemple, il peut s'agir de la position et de la vitesse en mécanique ; de la pression, de la température et du volume en thermodynamique ; et, en mécanique quantique, on les appelle des quantités observables. Du point de vue physique, si aucune quantité mesurable ne permet de distinguer deux états du système, alors ces états doivent être considérés comme égaux. De manière équivalente, un état du système est entièrement déterminé par les valeurs de ces quantités dans cet état. On souhaite alors décrire l'espace de tous les états possibles du système, que l'on peut appeler l'*espace des états*, ainsi que la manière dont ces états évoluent, que l'on peut appeler sa *dynamique*, afin de comprendre et de prédire le comportement du système.

Pour atteindre cet objectif, puisque les états sont déterminés par les valeurs de ces quantités, on peut d'abord former un *espace ambiant* plus général, dont les points sont tous les choix possibles de valeurs de ces quantités, qu'il existe ou non un état effectif du système prenant ces valeurs. Par exemple, cet espace ambiant peut être l'espace des phases en mécanique classique. À l'intérieur de cet espace abstrait, les quantités sont donc complètement indépendantes les unes des autres, ce qui ne reflète pas la situation du

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[1] Nous nous attendons à ce que les résultats de cette thèse s'étendent en caractéristique positive. Toutefois, certains d'entre eux demanderont des reformulations, en particulier en ce qui concerne la distinction entre feuilletages dérivés et feuilletages infinitésimaux, qui ne coïncident qu'en caractéristique nulle.

système physique réel ; en ce sens, l'espace ambiant est trop grand. Il faut donc imposer des relations entre ces quantités sur l'espace ambiant, souvent sous la forme de relations algébriques et d'équations différentielles, obtenues par l'étude physique du système. Plus concrètement, cela signifie que l'on définira de nouvelles quantités à partir des quantités mesurables, telles que l'énergie cinétique et l'énergie potentielle en mécanique, ou l'entropie en thermodynamique ; puis que l'on imposera des lois sur ces quantités, telles que les lois de conservation de l'énergie dans les systèmes conservatifs. Ces lois n'imposent pas toujours les valeurs exactes de ces fonctions, mais décrivent la manière dont leurs valeurs varient lorsque varient les valeurs des quantités du système. En définitive, cela impose des contraintes sur la manière dont le système peut évoluer, c'est-à-dire fournit de l'information sur sa dynamique. Cela suggère en particulier de partitionner l'espace ambiant en fonction des valeurs de ces fonctions.

Le choix d'une partition sur un espace s'appelle un *feuilletage*, et un morceau de cette partition s'appelle une *feuille*. En ce sens, les feuilletages peuvent apparaître en physique comme un moyen d'encoder les relations entre les quantités d'un système donné qui sont contraintes par la physique du système. Selon le système étudié et le type de relations imposées, on peut voir deux manières complémentaires dont la dynamique du système peut interagir avec le feuilletage. Parfois, des points de l'espace ambiant qui prennent les mêmes valeurs sur ces fonctions doivent être identifiés comme encodant le même état du système (c'est le cas par exemple en théorie de jauge) ; à l'inverse, il arrive que des points de l'espace ambiant qui prennent les mêmes valeurs sur ces fonctions correspondent à des états distincts du système, dont l'un peut être atteint à partir de l'autre au cours de l'évolution du système (c'est le cas par exemple en mécanique lagrangienne et hamiltonienne). Dans le premier cas, cela signifie que les états du système passent d'une feuille à l'autre au cours de leur évolution ; on dit alors que la dynamique est *transverse* au feuilletage. Dans le second cas, leur évolution est contrainte à l'intérieur des feuilles ; on dit alors que la dynamique est *longitudinale* ou *tangentielle* au feuilletage.

En pratique, la seule donnée d'un feuilletage ne suffit pas à encoder la dynamique du système, mais elle sert à esquisser le véritable espace des états à l'intérieur de l'espace ambiant. À ce titre, la donnée de l'espace ambiant et d'un choix physiquement motivé de feuilletage sur cet espace ambiant constitue une manière de présenter l'espace des états du système. Deux points importants doivent toutefois être soulignés : d'abord, en général, ni l'espace ambiant muni de son feuilletage, ni son espace des feuilles ne sont équivalents à l'espace des états ; ils n'en donnent qu'une présentation, au sens où il existe des moyens explicites ou conjecturaux de construire l'espace des états effectif à partir de ces données ; ensuite, selon le système étudié, il peut exister différentes manières de présenter le même espace des états, et donc différents choix possibles d'espace ambiant de quantités ou de relations entre ces quantités (c'est-à-dire de feuilletages). Ces deux points s'appliquent par exemple à la quantification géométrique, où la question est de savoir comment utiliser un espace ambiant classique et son feuilletage comme données d'entrée pour produire en sortie un espace quantique des états, ce qui correspond au premier point ci-dessus ; de plus, différents choix de feuilletages, appelés polarisations dans ce cadre, peuvent parfois conduire à la formation du même espace des états quantiques, ce qui correspond au second point ci-dessus.

Concentrons-nous maintenant sur la théorie mathématique des feuilletages et présentons nos résultats.

1.2 FEUILLETAGES ET RÉSULTATS PRINCIPAUX

En géométrie différentielle classique, un feuilletage sur une variété différentielle M est moralement la donnée d'une partition de M en sous-variétés, appelées les "feuilles" du feuilletage. Étant donné un tel feuilletage F et un point $x \in M$, on peut définir le sous-espace vectoriel $T_xF \subset T_xM$ des vecteurs tangents en x à $F_x \subset M$, où F_x est l'unique feuille de F contenant x. Pris collectivement, ils forment un sous-fibré $TF \subset TM$ du fibré tangent de M, appelé le fibré tangent du feuilletage, qui est de plus stable par le crochet de Lie. Par dualité, on obtient au contraire un quotient de l'algèbre de de Rham $\Omega^\bullet(M) \to \Omega^\bullet(F)$ qui hérite d'une différentielle interne provenant de la différentielle de de Rham, et $\Omega^1(F)$ s'appelle le complexe cotangent du feuilletage. De plus, lorsqu'elles sont correctement définies et dans le cas lisse, toutes ces descriptions des feuilletages se révèlent équivalentes. Une référence sur le sujet couvrant ces différentes définitions et leur équivalence est [MM03].

En géométrie algébrique classique, ce dernier point de vue a été utilisé pour définir les feuilletages algébriques, qui sont définis comme des idéaux différentiels du complexe cotangent. Cela permet en particulier de considérer des feuilletages sur des variétés singulières, ce qui n'était déjà pas possible dans le cadre différentiel, mais ils sont difficiles à construire et ne se transportent pas souvent le long des morphismes. Une référence sur les applications modernes de la théorie des feuilletages en géométrie algébrique est [CMP16].

En géométrie dérivée, nous considérons au contraire une notion de feuilletage dérivé qui est moins rigide et peut ainsi exister dans un éventail plus large de contextes, comme étudié dans [TV25a]. Plus précisément, dans cette thèse, nous démontrons le théorème principal suivant.

Théorème 1.2.1 (Théorème 6.2.4)**.** *Soit S un champ dérivé de base en caractéristique 0. Soit X un champ dérivé sur S dont le morphisme structural est schématique-propre, plat et localement d'intersection complète, et soit Y un champ dérivé de Deligne-Mumford relatif à S, localement de présentation finie, muni d'un feuilletage dérivé $\mathcal{F}$ relatif à S. Alors, en supposant que le champ dérivé des morphismes $\underline{\mathbf{Map}}_S(X,Y)$ soit un champ dérivé d'Artin relatif localement de présentation finie sur S, celui-ci hérite d'un feuilletage dérivé $\mathcal{F}_{\mathbf{Map}}$ relatif à S.*

Remarque 1.2.2. Notons qu'en comparaison, dans le cas différentiel, l'énoncé du théorème 1.2.1 demanderait déjà des clarifications quant à sa signification, puisque $\mathbf{Map}(X, Y)$ est un objet de dimension infinie. De plus, même dans le cas de la géométrie algébrique classique, où la signification d'un tel énoncé est claire, sa conclusion serait fausse dans cette généralité sans les outils de la géométrie dérivée et exigerait des hypothèses beaucoup plus fortes, car cet objet serait alors très singulier, son algèbre de de Rham extrêmement difficile à calculer, et non bornée.

En outre, comme en théorie classique des feuilletages, les feuilletages dérivés admettent des complexes tangents, et le complexe tangent du feuilletage dérivé induit sur $\underline{\mathbf{Map}}_S(X, Y)$ donné par le théorème ci-dessus admet la description suivante :

Proposition 1.2.3 (Proposition 6.2.5)**.** *Dans le contexte du théorème 1.2.1, notons* $\mathbb{T}_{\mathcal{F}}$ *(resp.* $\mathbb{T}_{\mathcal{F},\mathbf{Map}}$*) le complexe tangent de* $\mathcal{F}$ *(resp. de* $\mathcal{F}_{\mathbf{Map}}$*), et* $\mathbb{T}_{X/S}$ *le complexe tangent de* X *relatif à* S*. Pour tout* S*-point de* $\underline{\mathbf{Map}}_S(X, Y)$*, correspondant à un morphisme* S*-linéaire* $g : X \to Y$*, il existe une équivalence canonique :*

$$\mathbb{T}_g\mathcal{F}_{\mathbf{Map}} \simeq \Gamma_S(X, g^*\mathbb{T}_{\mathcal{F}})$$

où $\Gamma_S : \mathrm{QCoh}(X) \to \mathrm{QCoh}(S)$ *désigne le foncteur des sections globales relatives à la base* S*, c'est-à-dire l'image directe des faisceaux quasi-cohérents le long du morphisme structural* $X \to S$*, qui redonne les sections globales usuelles lorsque* $S = \mathrm{Spec}(k)$*. Plus généralement, le complexe tangent de* $\mathcal{F}_{\mathbf{Map}}$ *lui-même est donné par :*

$$\mathbb{T}_{\mathcal{F},\mathbf{Map}} \simeq \pi_* ev^* \mathbb{T}_{\mathcal{F}}$$

où $ev : X \times_S \underline{\mathbf{Map}}_S(X, Y) \to Y$ *est l'évaluation, et* $\pi : X \times_S \underline{\mathbf{Map}}_S(X, Y) \to \underline{\mathbf{Map}}_S(X, Y)$ *est la projection.*

Intuitivement, cette propriété signifie qu'une feuille L de $\mathcal{F}_{\mathbf{Map}}$ est moralement la donnée d'une famille de morphismes $(g_l : X \to Y)_{l\in L}$ telle que, pour tout $x \in X$, la famille $(g_l(x))_{l\in L}$ soit une feuille de $\mathcal{F}$.

Donnons à présent d'importants exemples d'applications de ce théorème, qui sont également démontrées dans cette thèse.

Exemple 1.2.4 (Voir la preuve dans le corollaire 8.1.2)**.** Soit $\mathbb{R}\mathcal{C}^{pre}_{g,n}$ l'enrichissement dérivé de la famille universelle des courbes de genre g à n points marqués, définie sur l'enrichissement dérivé de l'espace de modules des courbes pré-stables de genre g à n points marqués $\mathbb{R}\mathbf{M}^{pre}_{g,n}$, et soit X un schéma projectif lisse muni d'un feuilletage dérivé. Dans ce cas, $\mathbb{R}\mathbf{M}^{pre}_{g,n}(X)$, défini comme $\underline{\mathbf{Map}}_{\mathbb{R}\mathbf{M}^{pre}_{g,n}}(\mathbb{R}\mathcal{C}^{pre}_{g,n}, X \times \mathbb{R}\mathbf{M}^{pre}_{g,n})$, hérite d'un feuilletage dérivé ; de plus, il contient comme sous-champ ouvert $\mathbb{R}\overline{\mathbf{M}}_{g,n}(X;\beta)$, qui est l'enrichissement dérivé de l'espace de modules des familles de courbes stables de genre g à n points marqués, de classe β, dans X, lequel hérite lui aussi d'un feuilletage dérivé par restriction. Cela fournit une construction plus systématique du feuilletage dérivé introduit dans [TV25b, Section 2.2], où seul le cas des feuilletages dérivés provenant de champs de vecteurs était traité, tandis que les résultats de cette thèse s'étendent à des feuilletages dérivés arbitraires, même lorsqu'ils ne proviennent ni de champs de vecteurs ni de morphismes. En particulier, on pourrait considérer un nouveau type d'invariants de Gromov-Witten, relatifs à ce feuilletage dérivé. Notons que, dans l'énoncé de cet exemple, nous nous sommes restreints au cas où X est une variété projective lisse précisément pour correspondre au cadre usuel de la théorie de Gromov-Witten ; toutefois, notre construction produirait un feuilletage pour des X plus généraux.

Exemple 1.2.5 (Voir la preuve dans le corollaire 8.2.2)**.** De même, soit Y un champ dérivé de Deligne-Mumford localement de présentation finie et muni d'un feuilletage dérivé. Nous pouvons former le champ dérivé $\mathbb{R}\mathbf{Hilb}^{lci}(Y)$, dont les S-points, pour S un schéma dérivé, sont les immersions fermées $i : Z \hookrightarrow Y$ où Z est un schéma dérivé sur S et $Z \to S$ est propre et localement d'intersection complète (lci) ; ainsi, $\mathbb{R}\mathbf{Hilb}^{lci}(Y)$ s'immerge canoniquement dans un champ des morphismes et hérite d'un feuilletage dérivé issu de Y. De plus, nous pouvons en particulier étudier les points intégraux de ce feuilletage dérivé, c'est-à-dire le lieu où le feuilletage est nul. Voir [CC09, Def. 1.9.2(1)] pour la définition originelle en géométrie différentielle. Dans notre cadre, nous nous attendons à ce que ces points intégraux correspondent aux sous-schémas dérivés fermés $Z \subset Y$ stables par le feuilletage dérivé sur Y, c'est-à-dire aux sous-schémas dérivés fermés des feuilles algébriques du feuilletage dérivé.

Pour davantage d'informations sur ces deux exemples, voir le chapitre 8. Des résultats analogues pourraient être obtenus pour divers espaces de modules dérivés, pourvu que leur objet universel soit schématique-propre, plat et localement d'intersection complète au-dessus de l'espace de modules dérivé considéré.

Afin de démontrer le théorème 1.2.1, nous construirons plus généralement une image directe de feuilletages dérivés, comme énoncé ci-dessous. Notons que, pour un morphisme de champs dérivés $Z \to X$, nous noterons $\mathcal{F}ol(Z/X)$ l'∞-catégorie des feuilletages dérivés définis sur Z relativement à X, et que, pour un morphisme de champs dérivés $f : X \to Y$, nous noterons $f_* : \mathrm{dSt}\,/X \to \mathrm{dSt}\,/Y$ le foncteur de poussé en avant, aussi appelé image directe ou restriction de Weil.

Théorème 1.2.6 (Théorème 6.2.2)**.** *Soit $Z \to X$ un morphisme de champs dérivés qui soit un champ dérivé de Deligne-Mumford relatif localement de présentation finie, et soit $f : X \to Y$ un morphisme de champs dérivés schématique-propre, plat et localement d'intersection complète. Supposons que $f_*Z \to Y$ soit un champ dérivé d'Artin relatif localement de présentation finie. Alors il existe un foncteur image directe :*

$$f_{*,\mathcal{F}ol} : \mathcal{F}ol(Z/X) \to \mathcal{F}ol(f_*Z/Y)$$

Afin de construire cette image directe, nous montrerons que $\mathcal{F}ol(Z/X)$ est équivalente à une ∞-catégorie de champs dérivés sur Z munis d'une structure supplémentaire (voir corollaire 5.3.25), et que le foncteur canonique image directe des champs dérivés f_* : $\mathrm{dSt}/X \to \mathrm{dSt}/Y$ préserve cette structure (voir le corollaire 6.1.12). De plus, le complexe tangent de l'image directe d'un feuilletage admet une description très semblable à celle du théorème 1.2.3, voir la remarque 6.2.6. Enfin, notons que le cas particulier suivant du théorème ci-dessus implique le théorème 1.2.1.

Exemple 1.2.7. Soit S un champ dérivé, soit X un champ dérivé sur S dont le morphisme structural est schématique-propre, plat et localement d'intersection complète, et soit F un champ dérivé de Deligne-Mumford relatif à S, localement de présentation finie, muni d'un feuilletage dérivé relatif à S. Soient $Y = S$ et $Z = X \times_S F$, et remarquons que $X \to Y$ est alors schématique-propre, plat et localement d'intersection complète, et que la projection $Z \to X$ est un champ dérivé de Deligne-Mumford relatif localement de présentation finie. De plus, il existe une équivalence canonique $f_*Z = \underline{\mathbf{Map}}_S(X, F)$ démontrée dans la preuve du théorème 6.2.4. En supposant que $\underline{\mathbf{Map}}_S(X, F)$ soit un champ dérivé d'Artin relatif sur S, celui-ci hérite d'un feuilletage dérivé par application du théorème ci-dessus.

Comme mentionné au début de ce chapitre, cette thèse définit également des structures symplectiques avec décalage sur les feuilletages dérivés, et démontre des théorèmes d'existence pour ces structures. Nous allons donc maintenant motiver et introduire ces objets eux aussi.

1.3 STRUCTURES SYMPLECTIQUES SUR LES FEUILLETAGES ET RÉSULTATS PRINCIPAUX

Revenons au cas d'un système physique, comme dans la première section de ce chapitre. Comme nous l'avons expliqué, la donnée d'un feuilletage sur l'espace ambiant peut servir de présentation de l'espace des états du système, mais elle n'encode pas la dynamique du système elle-même, bien qu'elle puisse interagir avec celle-ci de façon transversale ou longitudinale. Pour encoder la dynamique, une structure supplémentaire est donc requise. Comme nous l'avons également mentionné dans la première section, la dynamique provient ultimement de fonctions définies à partir des quantités mesurables du système, ainsi que de lois physiques portant sur ces fonctions, qui contraignent les évolutions possibles du système jusqu'à les déterminer. Toutefois, du point de vue mathématique, il faut un outil permettant de traduire les données de ces fonctions en équations qui déterminent l'évolution du système. Un tel outil est fourni par la donnée d'une *structure symplectique* sur l'espace ambiant, ou plus généralement par une *structure de Poisson*, et tous les systèmes connus de la physique moderne peuvent être plongés dans des géométries symplectiques ou (presque-)de Poisson. Il est important de noter que d'autres outils existent et sont, dans certains contextes, plus intrinsèques, comme les géométries de contact, en particulier dans le cas des systèmes statistiques présentant un comportement irréversible, tel que la production d'entropie. Notons également que la dynamique du système n'est alors pas encodée par la structure symplectique ou de Poisson elle-même, mais par le choix d'une fonction H appelée un *hamiltonien*, définie sur l'espace ambiant des quantités du système, et que la structure symplectique ou de Poisson est l'outil qui transforme la donnée de H en équations d'évolution du système.

En géométrie différentielle classique, la donnée d'une forme symplectique sur une variété M est moralement la donnée du choix d'un isomorphisme entre TM et T^*M, les fibrés tangent et cotangent de M. Ainsi, étant donné un hamiltonien $H : M \to \mathbb{R}$ comme ci-dessus, celui-ci détermine d'abord, par différentiation, une 1-forme $dH \in T^*M$, qui ne peut être identifiée à un champ de vecteurs $X_H \in TM$ qu'à l'aide de la forme symplectique. Le flot de ce champ de vecteurs fournit alors les trajectoires du système physique. Notons que la donnée d'une forme symplectique est habituellement donnée sous la forme d'une forme bilinéaire ω sur TM, qui est fermée, antisymétrique et non dégénérée ; l'isomorphisme $TM \xrightarrow{\sim} T^*M$ est alors donné par $x \mapsto \omega(x, -)$ (ou $\omega(-, x)$, à une convention de signe près). On peut montrer que, pour qu'une variété M admette une forme symplectique, il faut qu'elle soit de dimension paire $2n$, et que, localement, la forme symplectique admette une description canonique, au sens où il existe des coordonnées $(p_1, ..., p_n, q_1, ..., q_n)$ dans lesquelles $\omega = \sum_i dp_i \wedge dq_i$, appelées *coordonnées de Darboux*. Dans ces coordonnées de Darboux, on peut voir les q_i comme des positions et les p_i comme des moments, et les équations du mouvement du système s'écrivent efficacement dans ces coordonnées. Si l'on fixe les q_i et que l'on regarde le sous-espace obtenu lorsque les p_i varient, on obtient une *sous-variété lagrangienne* à l'intérieur de l'espace ambiant. Étudier son évolution sous la dynamique du système décrit comment un état à position fixée peut évoluer en fonction de sa vitesse initiale dans toutes les directions possibles. Plus généralement, on peut vouloir spécifier globalement une manière de distinguer positions et moments dans une variété symplectique ; on est alors conduit à considérer un feuilletage de la variété symplectique dont chaque feuille est une sous-variété lagrangienne : une telle donnée s'appelle un *feuilletage lagrangien*. Le choix d'un feuilletage lagrangien dans une variété symplectique peut ainsi être vu comme une manière de présenter l'espace des états d'un système physique sous la forme d'une décomposition en positions et moments.

Comme mentionné ci-dessus, il existe une donnée plus générale : celle d'une structure de Poisson. Elle est moralement la donnée d'une application $\pi : T^*M \to TM$, dont on ne requiert plus qu'elle soit un isomorphisme, contrairement au cas des structures symplectiques. Plus précisément, il s'agit de la donnée d'un bivecteur $\pi \in \Gamma(\wedge^2 TM)$ satisfaisant $[\pi, \pi] = 0$, où $[-, -]$ désigne le crochet de Schouten-Nijenhuis, qui généralise le crochet de Lie des champs de vecteurs. Dans ce cas, étant donné un hamiltonien $H : M \to \mathbb{R}$, le champ de vecteurs associé est $X_H = \pi(dH)$. En fait, nous pouvons préciser plus exactement dans quelle mesure les structures de Poisson généralisent les structures symplectiques : dans [Wei83], Weinstein démontre que la donnée d'une structure de Poisson sur une variété M est équivalente à la donnée d'un feuilletage sur M et d'une structure symplectique sur chaque feuille du feuilletage, variant de manière lisse lorsqu'on passe d'une feuille à l'autre (c'est-à-dire transversalement au feuilletage). Une telle structure est appelée un *feuilletage symplectique* sur M. Du point de vue du système physique représenté par la variété de Poisson, la dynamique du système est contenue dans les feuilles du feuilletage symplectique, c'est-à-dire qu'elle est longitudinale par rapport à ce feuilletage ; à l'intérieur d'une feuille, elle est gouvernée par la structure symplectique portée par cette feuille. Cela suggère d'étudier non seulement les structures symplectiques, mais aussi les structures de Poisson et les feuilletages symplectiques. Chaque feuille peut alors être munie d'un feuilletage lagrangien supplémentaire comme ci-dessus, variant de manière lisse lorsqu'on change de feuille dans le feuilletage symplectique. Cela permet une décomposition plus fine de l'espace des états, donnée localement par des coordonnées permettant une description effective de la dynamique du système comme dans le cas symplectique ; en géométrie de Poisson, ces coordonnées sont appelées *coordonnées de Darboux-Weinstein.*

En géométrie dérivée, tandis que les structures de Poisson ont été définies et étudiées dans [CPTVV], que les structures symplectiques ont été définies et étudiées dans [PTVV], et que les feuilletages ont été définis et étudiés dans [TV25a], la notion de feuilletage symplectique n'avait encore jamais été définie. Dans cette thèse, étant donné un feuilletage dérivé $\mathcal{F}$ de tangent $\mathbb{T}_{\mathcal{F}}$ et de cotangent $\mathbb{L}_{\mathcal{F}}$, nous définissons, par analogie avec les définitions de [PTVV], une forme symplectique n-décalée sur $\mathcal{F}$ comme étant la donnée d'une 2-forme non dégénérée $\omega : \mathbb{T}_{\mathcal{F}} \xrightarrow{\sim} \mathbb{L}_{\mathcal{F}}[n]$, qui est de plus fermée pour la différentielle mixte de $\mathcal{F}$ (laquelle, comme dans le cas des structures symplectiques n-décalées, constitue une structure supplémentaire et non une simple condition, contrairement au cas classique). Cela définit la notion de *feuilletage dérivé symplectique n-décalé.* Dans une direction complémentaire, étant donné un champ dérivé X muni d'une structure symplectique n-décalée ω, nous définissons également la notion de *feuilletage dérivé lagrangien.* Étant donné un feuilletage dérivé $\mathcal{F}$ sur X, de tangent $\mathbb{T}_{\mathcal{F}}$, de cotangent $\mathbb{L}_{\mathcal{F}}$, et de conormal $\mathcal{N}^*_{\mathcal{F}} := \mathrm{fib}(\mathbb{L}_X \to \mathbb{L}_{\mathcal{F}})$, la donnée d'une homotopie vers 0 de la restriction de ω à $\mathbb{T}_{\mathcal{F}}$ est appelée une *structure isotrope sur $\mathcal{F}$* et induit une application canonique $\mathbb{T}_{\mathcal{F}} \to \mathcal{N}^*_{\mathcal{F}}[n]$. On dit alors que $\mathcal{F}$ est un feuilletage dérivé lagrangien si cette application est une équivalence.

À partir de ces deux définitions, nous démontrons les théorèmes d'existence suivants pour ces structures :

Théorème 1.3.1 (Théorème 7.3.4). *Soit $Z \to X$ un morphisme de champs dérivés qui soit localement de présentation finie et admette un cotangent relatif $\mathbb{L}_{Z/X}$, soit $\omega \in \mathbb{S}ymp(Z/X, n)$ une forme symplectique n-décalée, et soient $\mathcal{F}, \mathcal{F}' \in \mathcal{F}ol(Z/X)$ deux feuilletages dérivés munis de structures lagrangiennes h et h'. Alors le feuilletage dérivé d'intersection $\mathcal{F}'' := \mathcal{F} \times_{Z/X} \mathcal{F}' \in \mathcal{F}ol(Z/X)$ hérite d'une forme symplectique $(n-1)$-décalée.*

Théorème 1.3.2 (Théorème 7.3.18)**.** *Soit $Z \to X$ un champ dérivé de Deligne-Mumford relatif localement de présentation finie, et soit $f : X \to Y$ un morphisme de champs dérivés schématique-propre, localement d'intersection complète et plat. Supposons que $f_*Z \to Y$ soit un champ dérivé d'Artin relatif localement de présentation finie. Soit f muni de $[X]$, une $\mathcal{O}$-orientation de degré d, et soit $\mathcal{F} \in \mathcal{F}ol(Z/X)$ un feuilletage dérivé sur Z relatif à X, muni d'une forme symplectique n-décalée $\omega \in \mathbb{S}ymp(\mathcal{F}, n)$. Alors le feuilletage dérivé image directe $f_*\mathcal{F} \in \mathcal{F}ol(f_*Z/Y)$ hérite d'une forme symplectique $(n-d)$-décalée $f_*\omega \in \mathbb{S}ymp(f_*\mathcal{F}, n-d)$.*

1.4 PLAN DE LA THÈSE

Pour conclure ce chapitre introductif, nous allons maintenant présenter le plan de cette thèse. Il est à noter que le chapitre 2 est une traduction anglaise du présent chapitre. Il faut également noter que les chapitres 4, 5, 6 et l'annexe A proviennent en grande partie de mon précédent article [Alf25].

Dans le chapitre 3, nous commencerons par motiver et introduire la géométrie algébrique dérivée. Ce chapitre contient une série de définitions et d'énoncés de propriétés sur les ∞-catégories, les ∞-topos et les ∞-catégories stables, puis sur les objets à la base de la géométrie algébrique dérivée, tels que les affines dérivés, les champs dérivés et les faisceaux quasi-cohérents. Ce chapitre vise à permettre à des lecteurs familiers avec la géométrie algébrique classique et avec la théorie des catégories de se faire une idée plus concrète de cette théorie.

Dans le chapitre 4, nous rappellerons d'abord les définitions algébriques des feuilletages dérivés, puis nous les reformulerons à l'aide d'objets géométriques, en revenant sur des résultats d'articles antérieurs. Les conclusions de cette section sont résumées dans la conclusion 4.3.8.

Dans le chapitre 5, cette description géométrique sera utilisée pour construire un foncteur de l'∞-catégorie des feuilletages dérivés vers une ∞-catégorie de champs dérivés munis d'une structure supplémentaire, comme résumé dans la conclusion 5.1.12. Nous montrerons ensuite que ce foncteur est pleinement fidèle, et décrirons explicitement son image essentielle, ce qui fournit une équivalence d'∞-catégories entre les feuilletages dérivés et cette image essentielle, comme énoncé dans le théorème 5.2.9 dans le cas affine, puis généralisé aux champs dérivés relatifs de Deligne-Mumford dans le corollaire 5.3.25. Notons que cette thèse contient une annexe A, qui consiste en une collection de résultats sur les conditions de Beck-Chevalley, précisément parce que ces conditions apparaissent dans les preuves de la section 5.2 de ce chapitre.

Dans le chapitre 6, nous montrerons que le foncteur image directe des champs dérivés au-dessus de la base vers les champs dérivés au-dessus du but préserve la structure supplémentaire des champs dérivés mentionnés ci-dessus, comme énoncé dans le corollaire 6.1.12.

Cela donnera à la fois le théorème 1.2.6 ci-dessus et, finalement, le théorème 1.2.1, ainsi que la description du complexe tangent des feuilletages dérivés ainsi obtenus dans le théorème 1.2.3.

Dans le chapitre 7, nous définirons des structures symplectiques avec décalage sur les feuilletages dérivés et démontrerons des théorèmes d'existence pour ces structures.

Dans le chapitre 8, nous démontrerons deux applications des résultats de cette thèse, mentionnées dans l'exemple 1.2.4 et l'exemple 1.2.5, respectivement pour la théorie de Gromov-Witten et pour les schémas de Hilbert.

Enfin, dans le chapitre 9, nous discuterons de travaux en cours et de perspectives futures dans le prolongement de cette thèse.

Usage d'IA générative : une IA générative a été utilisée à des fins bibliographiques, de résumé de textes personnels pour certains textes introductifs d'autres chapitres ultérieurs, et de corrections d'orthographe, de grammaire et de typographie.

CHAPTER 2

INTRODUCTION

In this thesis, we will work in derived algebraic geometry in characteristic zero.[1] We will prove existence theorems for derived foliations, define shifted symplectic structures on them, and also prove existence theorems for these structures. Derived algebraic geometry was introduced by Bertrand Toën and Gabriele Vezzosi in [HAGI; HAGII], and by Jacob Lurie in [LuriePhD]. Foliations in this context were introduced and studied in a series of papers by Bertrand Toën and Gabriele Vezzosi, [TV23a] and [TV20], and later on in their book [TV25a]; while shifted symplectic structures were defined and studied in [PTVV] and [CPTVV]. Let us now motivate and introduce these objects.

2.1 A PHYSICAL MOTIVATION TO FOLIATIONS

In physics, one studies a physical system by inspecting the different states that the system can be in. These states are distinguished from each other by measurable quantities of the system: for instance these can be position and velocity in mechanics; they can be pressure, temperature and volume in thermodynamics; and they are called observable quantities in quantum mechanics. From a physical point of view, if no measurable quantity can distinguish two states of the system, then these states are to be considered equal. Equivalently, a state of the system is entirely determined by the values of these quantities in this state. One then wishes to describe the space of all possible states of the system, which can be called the *state space*, and how these states evolve, which can be called its *dynamics*, in order to understand and predict the behaviour of the system.

To achieve this goal, since states are determined by the values of the quantities, one can thus form first a more general *ambient space*, where the points are all possible choices of values of these quantities, whether or not there exists an actual state of the system which takes these values. For example, this ambient space can be the phase space in classical mechanics. Inside this abstract space, the quantities are thus completely independent from each other, which does not reflect the situation on the actual physical system, and in this sense the ambient space is too large. One thus has to impose relations between these quantities on the ambient space, often in the form of algebraic relations and differential equations, that are obtained by the physical study of the system. More concretely, this

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[1] We expect results from this thesis to extend to positive characteristics. However, some of them will require reformulations, particularly in the distinction between derived foliations and infinitesimal foliations which only coincide in characteristic zero.

means that one will define new quantities in terms of the measurable ones, such as kinetic and potential energy in mechanics, or entropy in thermodynamics; and impose laws on these quantities, such as energy conservation laws in conservative systems. These laws may not always impose the exact values of these functions, but describe how the values of these functions change when the values of the quantities of the system change. Ultimately, this imposes constraints on how the system can evolve, i.e. information on its dynamics. This suggests in particular to partition the ambient space in terms of the values of these functions.

A choice of a partition of a space is called a *foliation*, and a piece of this partition is called a *leaf*. In this sense, foliations in physics can arise as a way to encode the relations between the quantities of a given system that are constrained by the physics of the system. Depending on the system that is studied and the type of relations that are imposed, we can see two complementary ways in which the dynamics of the system can interact with the foliation. Sometimes, points of the ambient space which take the same values on these functions are to be identified as encoding the same state of the system (this is the case for example in gauge theory), or on the contrary sometimes points of the ambient space which take the same values on those functions may correspond to distinct states of the system where one can be reached from the other as the system evolves (this is the case for example in Lagrangian and Hamiltonian mechanics). In the first case, this means that the states of the system pass through different leaves as they evolve, in which case the dynamics is said to be *transverse* to the foliation; while in the second case, their evolution is constrained within the leaves, in which case the dynamics is said to be *longitudinal* or *tangential* to the foliation.

In practice, the sole data of a foliation is not enough to encode the dynamics of the system, but serves to outline the actual state space within the ambient space. As such, the data of the ambient space and a physically informed choice of a foliation on this ambient space serve as a way to give a presentation of the state space of the system. However, two important points should be noted: first, in general neither the ambient space equipped with its foliation nor its leaf space are equivalent to the state space, they only give a presentation of it in the sense that there are explicit or conjectural ways to construct the actual state space from this data; second, depending on the system that is studied, there can be different ways to present the same state space, and therefore different choices of ambient space of quantities or relations between those quantities (i.e. foliations) that can be available. Both points apply for instance in geometric quantization, where the question is how to use a classical ambient space and its foliation as input data to produce a quantum state space as output, corresponding to the first point above; and moreover where different choices of foliations, called polarizations in this setting, may sometimes lead to the formation of the same space of quantum states, corresponding to the second point above.

Let us now focus on the mathematical theory of foliations and present our results.

2.2 Foliations and Main Results

In classical differential geometry, a foliation on a differential manifold M is morally the data of a partition of M into sub-manifolds, called the "leaves" of the foliation. Given such a foliation F and a point $x \in M$, one can define the sub-vector space $T_xF \subset T_xM$ of tangent vectors to x in $F_x \subset M$, where F_x is the only leaf of F containing x. Collectively they form a sub-bundle $TF \subset TM$ of the tangent bundle of M, called the tangent bundle of the foliation, which is moreover stable by the Lie bracket. By dualizing, one obtains instead a quotient of the de Rham algebra $\Omega^\bullet(M) \to \Omega^\bullet(F)$ which inherits an internal differential from the de Rham differential, and $\Omega^1(F)$ is called the cotangent complex of the foliation. Moreover, when properly defined and in the smooth case, all these descriptions of foliations turn out to be equivalent to each other. A reference on the subject covering these different definitions and their equivalence is [MM03].

In classical algebraic geometry, the latter point of view has been used to define algebraic foliations, which are defined as differential ideals of the cotangent complex. This allows in particular to consider foliations on singular varieties which was already not possible in the differential case, but they are difficult to construct and cannot often be transported along morphisms. A reference on modern applications of the theory of foliations in algebraic geometry is [CMP16].

In derived geometry, we consider instead a notion of derived foliation which is less rigid and can thus exist in a wider range of contexts, as studied in [TV25a]. Most notably, in this thesis we will prove the following main theorem.

Theorem 2.2.1 (Theorem 6.2.4)**.** *Let S be a base derived stack in characteristic 0. Let X be a proper-schematic, flat, and local complete intersection derived stack over S, and Y be a relative derived Deligne-Mumford stack locally of finite presentation over S, equipped with a derived foliation $\mathcal{F}$ relative to S. Then, assuming the derived mapping stack $\underline{\mathbf{Map}}_S(X,Y)$ is a relative derived Artin stack locally of finite presentation over S, it inherits a derived foliation $\mathcal{F}_{\mathbf{Map}}$ relative to S.*

Remark 2.2.2. Note that in comparison, in the differential case, the statement of theorem 2.2.1 would already require clarifications on its meaning because $\mathbf{Map}(X,Y)$ is an infinite dimensional object. Moreover, even in the case of classical algebraic geometry where the meaning of such a statement is clear, without the tools of derived geometry its conclusion would be false in this generality and require much stronger hypotheses, because this object would be very singular and its de Rham algebra extremely difficult to compute and not bounded.

Furthermore, as in classical foliation theory, derived foliations admit tangent complexes, and the tangent complex of the induced derived foliation on $\underline{\mathbf{Map}}_S(X,Y)$ stated by the above theorem has the following description:

Proposition 2.2.3 (Proposition 6.2.5)**.** *In the context of theorem 2.2.1 (6.2.4), let* $\mathbb{T}_{\mathcal{F}}$ *(resp.* $\mathbb{T}_{\mathcal{F},\mathbf{Map}}$*) denote the tangent complex of* $\mathcal{F}$ *(resp.* $\mathcal{F}_{\mathbf{Map}}$*), and* $\mathbb{T}_{X/S}$ *the tangent complex of* X *relative to* S*. For any* S*-point of* $\underline{\mathbf{Map}}_S(X,Y)$*, which corresponds to an* S*-linear map* $g : X \to Y$*, there is a canonical equivalence:*

$$\mathbb{T}_g\mathcal{F}_{\mathbf{Map}} \simeq \Gamma_S(X, g^*\mathbb{T}_{\mathcal{F}})$$

where $\Gamma_S : \mathrm{QCoh}(X) \to \mathrm{QCoh}(S)$ *is the global-sections functor relative to the base* S*, i.e. the push-forward of quasi-coherent sheaves along the structural map* $X \to S$*, which recovers the usual global sections when* $S = \mathrm{Spec}(k)$*. More generally, the tangent complex of* $\mathcal{F}_{\mathbf{Map}}$ *itself is given by:*

$$\mathbb{T}_{\mathcal{F},\mathbf{Map}} \simeq \pi_* ev^* \mathbb{T}_{\mathcal{F}}$$

where $ev : X \times_S \underline{\mathbf{Map}}_S(X,Y) \to Y$ *is the evaluation, and* $\pi : X \times_S \underline{\mathbf{Map}}_S(X,Y) \to \underline{\mathbf{Map}}_S(X,Y)$ *is the projection*

Intuitively, this property means that a leaf L of $\mathcal{F}_{\mathbf{Map}}$ is morally the data of a family of maps $(g_l : X \to Y)_{l\in L}$ such that, for any $x \in X$, the family $(g_l(x))_{l\in L}$ is a leaf of $\mathcal{F}$.

Let us now give some important example applications of this theorem, which are also proved in this thesis.

Example 2.2.4 (See proof in corollary 8.1.2)**.** Let $\mathbb{R}\mathcal{C}^{pre}_{g,n}$ be the derived enhancement of the universal family of curves of genus g with n marked points, defined over the derived enhancement of the moduli stack of pre-stable curves of genus g with n marked points $\mathbb{R}\mathbf{M}^{pre}_{g,n}$, and X be a smooth projective scheme equipped with a derived foliation. In this case, $\mathbb{R}\mathbf{M}^{pre}_{g,n}(X)$, defined as $\underline{\mathbf{Map}}_{\mathbb{R}\mathbf{M}^{pre}_{g,n}}(\mathbb{R}\mathcal{C}^{pre}_{g,n}, X \times \mathbb{R}\mathbf{M}^{pre}_{g,n})$, inherits a derived foliation, and moreover it contains $\mathbb{R}\overline{\mathbf{M}}_{g,n}(X;\beta)$ as an open substack, which is the derived enhancement of the moduli stack of families of stable curves of genus g with n marked points of class β on X, which also inherits a derived foliation by restriction. This provides a more systematic construction of the derived foliation introduced in [TV25b, Section 2.2], in which only the case of derived foliations arising from vector fields was treated, whereas the results of this thesis extend to arbitrary derived foliations even when they do not arise from vector fields or maps. In particular, one could consider a new kind of Gromov-Witten invariants, relative to this derived foliation. Note that in the statement of this example, we restricted ourselves to X being a smooth projective variety precisely to match the usual setting of Gromov-Witten theory, however our construction would yield a foliation for more general X.

Example 2.2.5 (See proof in corollary 8.2.2)**.** Similarly, let Y be a derived Deligne-Mumford stack locally of finite presentation equipped with a derived foliation. We can form the derived stack $\mathbb{R}\mathbf{Hilb}^{lci}(Y)$, whose S-points for S a derived scheme are closed immersions $i : Z \hookrightarrow Y$ where Z is a derived scheme over S and $Z \to S$ is proper and local complete intersection (lci), thus $\mathbb{R}\mathbf{Hilb}^{lci}(Y)$ is canonically embedded in a mapping stack, and inherits a derived foliation from Y. Moreover, we can study in particular the integral points of this derived foliation, i.e. the locus where the foliation is 0. See [CC09, Def. 1.9.2(1)] for the original definition in differential geometry. In our setting, we expect these integral points to correspond to closed derived subschemes $Z \subset Y$ that are stable by the derived foliation on Y, i.e. closed derived subschemes of algebraic leaves of the derived foliation.

For more information on both of these examples, see chapter 8. Similar results could be obtained for various derived moduli stacks, provided the universal object is proper-schematic, flat, and local complete intersection over the derived moduli stack in question.

In order to prove theorem 2.2.1, we will more generally construct a push-forward of derived foliations, as stated next. Note that for a map of derived stacks $Z \to X$, we will denote by $\mathcal{F}ol(Z/X)$ the ∞-category of derived foliations defined on Z relative to X, and for a map of derived stacks $f : X \to Y$, we will denote by $f_* : \mathrm{dSt}/X \to \mathrm{dSt}/Y$ the push-forward functor, also called direct image or Weil restriction.

Theorem 2.2.6 (Theorem 6.2.2)**.** *Let $Z \to X$ be a map of derived stacks which is a relative derived Deligne-Mumford stack locally of finite presentation, and $f : X \to Y$ a proper-schematic, flat, and local complete intersection map of derived stacks. Assume that $f_*Z \to Y$ is a relative derived Artin stack locally of finite presentation. Then, there is a push-forward functor:*

$$f_{*,\mathcal{F}ol} : \mathcal{F}ol(Z/X) \to \mathcal{F}ol(f_*Z/Y)$$

In order to construct this push-forward, we will show that $\mathcal{F}ol(Z/X)$ is equivalent to an ∞-category of derived stacks over Z equipped with extra structure (see corollary 5.3.25), and that the canonical push-forward functor of derived stacks $f_* : \mathrm{dSt}/X \to \mathrm{dSt}/Y$ preserves this structure (see corollary 6.1.12). Moreover, the tangent complex of the push-forward of a foliation admits a description very similar to that of proposition 2.2.3, see remark 6.2.6. Finally, note that the following special case of the above theorem yields the main theorem 2.2.1.

Example 2.2.7. Let S be a derived stack, X be a proper-schematic, flat, and local complete intersection derived stack over S, and F be a relative derived Deligne-Mumford stack locally of finite presentation over S equipped with a derived foliation relative to S. Let $Y = S$ and $Z = X \times_S F$, and note that $X \to Y$ is thus proper-schematic, flat, and local complete intersection, and that the projection $Z \to X$ is a relative derived Deligne-Mumford stack locally of finite presentation. Moreover, there is a canonical equivalence $f_*Z = \underline{\mathbf{Map}}_S(X, F)$ shown in the proof of theorem 6.2.4. Assuming $\underline{\mathbf{Map}}_S(X, F)$ is a relative derived Artin stack over S, the latter inherits a derived foliation by application of the above theorem.

As mentioned at the beginning of this chapter, this thesis also defines shifted symplectic structures on derived foliations, and proves existence theorems for these structures. As such, we will now motivate and introduce these objects as well.

2.3 Symplectic Structures on Foliations and Main Results

Let us come back to the case of a physical system, as in the first section of this chapter. As explained, the data of a foliation on the ambient space can serve as a presentation of the state space of the system, but it does not encode the dynamics of the system itself, although it can interact with it transversely or longitudinally. To encode the dynamics, extra structure is thus required. As also mentioned in the first section, the dynamics ultimately comes from functions defined in terms of the measurable quantities of the system, and physical laws on these functions that constrain the possible evolutions of the system to the point of determining them. However, from a mathematical point of view, a tool is required to translate the data of these functions into equations that determine the evolution of the system. Such a tool is given by the data of a *symplectic structure* on the ambient space, or more generally by a *Poisson structure*, and all known systems of modern physics can be embedded in symplectic or (almost-)Poisson geometries. It is important to note that other tools exist and are more intrinsic in some contexts, such as contact geometries, particularly in the case of statistical systems that exhibit irreversible behaviour, such as entropy production. Also note that the dynamics of the system is then not encoded by the symplectic or Poisson structure itself, but by the choice of a function H called a *Hamiltonian* defined on the ambient space of quantities of the system, and the symplectic or Poisson structure is the tool that turns the data of H into equations of evolution of the system.

In classical differential geometry, the data of a symplectic form on a manifold M is morally the data of a choice of isomorphism between TM and T^*M, the tangent and cotangent bundles on M. As such, given a Hamiltonian $H : M \to \mathbb{R}$ as above, it first determines a cotangent form $dH \in T^*M$ by differentiation, which can only be identified with a vector field $X_H \in TM$ using the symplectic form. The flow of this vector field then gives the trajectories of the physical system. Note that the data of a symplectic form is usually given in the form of a bilinear form ω on TM that is closed, skew-symmetric and non-degenerate, so the isomorphism $TM \xrightarrow{\sim} T^*M$ is given by $x \mapsto \omega(x, -)$ (or $\omega(-, x)$ up to sign conventions). One can show that, for a manifold M to have a symplectic form, it is required that M has even dimension $2n$, and locally the symplectic form has a canonical description in the sense that there exist coordinates $(p_1, ..., p_n, q_1, ..., q_n)$ for which $\omega = \sum_i dp_i \wedge dq_i$, which are called *Darboux coordinates.* In Darboux coordinates, one can think of the q_i as positions and the p_i as momenta, and the equations of motion of the system can be written efficiently in these coordinates. If one fixes the q_i and looks at the subspace obtained when the p_i vary, one obtains a *Lagrangian submanifold* within the ambient space. Studying its evolution under the dynamics of the system describes how a state at a fixed position can evolve depending on its initial velocity in all possible directions. More generally, one may wish to specify a way to distinguish position and momenta globally in a symplectic manifold, in which case one is led to consider a foliation of the symplectic manifold in which each leaf is a Lagrangian submanifold: this is called a *Lagrangian foliation.* A choice of Lagrangian foliation in a symplectic manifold can thus be thought of as a way to present the state space of a physical system as a decomposition in position and momenta.

As mentioned above, there is a more general notion of Poisson structure. It is given instead morally as the data of a map $\pi : T^*M \to TM$, which is not required to be an isomorphism contrary to the case of symplectic structures. More precisely, it is the data of a bivector $\pi \in \Gamma(\wedge^2 TM)$, which satisfies $[\pi, \pi] = 0$, where $[-,-]$ is the Schouten–Nijenhuis bracket generalizing the Lie bracket of vector fields. In this case, given a Hamiltonian $H : M \to \mathbb{R}$, the associated vector field is $X_H = \pi(dH)$. In fact, we can state more precisely how much Poisson structures generalize symplectic structures: in [Wei83], Weinstein proves that the data of a Poisson structure on a manifold M is equivalent to the data of a foliation on M and a symplectic structure on each leaf of the foliation, varying smoothly as one changes leaves (i.e. transversally to the foliation). This structure is called a *symplectic foliation* on M. From the point of view of the physical system that the Poisson manifold represents, the dynamics of the system will be contained inside the leaves of the symplectic foliation, i.e. it is longitudinal to this foliation, and inside a leaf it will be governed by the symplectic structure on this leaf. This suggests studying not only symplectic structures, but also Poisson structures and symplectic foliations. Each leaf may then be equipped with a further Lagrangian foliation as above, varying in a smooth way as one changes leaves in the symplectic foliation. This allows a further decomposition of the state space, which is given locally in coordinates that allow an effective description of the dynamics of the system as in the symplectic case, which are called *Darboux-Weinstein coordinates* in Poisson geometry.

In derived geometry, while Poisson structures have been defined and studied in [CPTVV], symplectic structures have been defined and studied in [PTVV], and foliations have been defined and studied in [TV25a], the notion of symplectic foliation has not been defined before. In this thesis, given a derived foliation $\mathcal{F}$ with tangent $\mathbb{T}_{\mathcal{F}}$ and cotangent $\mathbb{L}_{\mathcal{F}}$, we define an n-shifted symplectic form on $\mathcal{F}$ by analogy with the definitions of [PTVV]: it is the data of a non-degenerate 2-form $\omega : \mathbb{T}_{\mathcal{F}} \xrightarrow{\sim} \mathbb{L}_{\mathcal{F}}[n]$, which is moreover closed with respect to the mixed differential of $\mathcal{F}$ (which, as in the case of n-shifted symplectic structures, is an extra structure and not merely a condition, contrary to the classical case). This defines the notion of *n-shifted symplectic derived foliations.* In a complementary direction, given a derived stack X equipped with an n-shifted symplectic structure ω, we also define the notion of *Lagrangian derived foliations.* Given a derived foliation $\mathcal{F}$ on

X, with tangent $\mathbb{T}_{\mathcal{F}}$, cotangent $\mathbb{L}_{\mathcal{F}}$, and conormal $\mathcal{N}^*_{\mathcal{F}} := \mathrm{fib}(\mathbb{L}_X \to \mathbb{L}_{\mathcal{F}})$, the data of a homotopy to 0 of the restriction of ω to $\mathbb{T}_{\mathcal{F}}$ is called an *isotropic structure on* $\mathcal{F}$, and induces a canonical map $\mathbb{T}_{\mathcal{F}} \to \mathcal{N}^*_{\mathcal{F}}[n]$. It is said to be a Lagrangian derived foliation if this map is an equivalence.

Given these two definitions, we prove the following existence theorems for these structures:

Theorem 2.3.1 (Theorem 7.3.4)**.** *Let $Z \to X$ be a map of derived stacks which is locally of finite presentation and admits a relative cotangent $\mathbb{L}_{Z/X}$, $\omega \in \mathbb{S}ymp(Z/X, n)$ an n-shifted symplectic form, and $\mathcal{F}, \mathcal{F}' \in \mathcal{F}ol(Z/X)$ be two derived foliations equipped with Lagrangian structures h and h'. Then the intersection derived foliation $\mathcal{F}'' := \mathcal{F} \times_{Z/X} \mathcal{F}' \in \mathcal{F}ol(Z/X)$ inherits an $(n-1)$-shifted symplectic form.*

Theorem 2.3.2 (Theorem 7.3.18)**.** *Let $Z \to X$ be a relative derived Deligne-Mumford stack locally of finite presentation, and $f : X \to Y$ a proper-schematic, local complete intersection, and flat morphism of derived stacks. Assume that $f_*Z \to Y$ is a relative derived Artin stack locally of finite presentation. Let f be equipped with $[X]$ an $\mathcal{O}$-orientation of degree d, and let $\mathcal{F} \in \mathcal{F}ol(Z/X)$ be a derived foliation on Z relative to X equipped with an n-shifted symplectic form $\omega \in \mathbb{S}ymp(\mathcal{F}, n)$. Then the push-forward derived foliation $f_*\mathcal{F} \in \mathcal{F}ol(f_*Z/Y)$ inherits an $(n-d)$-shifted symplectic form $f_*\omega \in \mathbb{S}ymp(f_*\mathcal{F}, n-d)$.*

2.4 Plan of the Thesis

To finish this introductory chapter, we will now outline the plan of this thesis. Note that chapter 1 is a French translation of this chapter. It should also be noted that substantial parts of the chapters 4, 5, 6 and the appendix A are adapted from my previous article [Alf25].

In chapter 3, we will start by motivating and introducing derived algebraic geometry. This chapter contains a series of definitions and statements of properties on ∞-categories, ∞-topoi and stable ∞-categories, and then on the objects at the foundation of derived algebraic geometry such as derived affines, derived stacks, and quasi-coherent sheaves. This chapter aims to allow readers familiar with classical algebraic geometry and category theory to get a more concrete sense of the theory.

In chapter 4, we will first recall the algebraic definitions of derived foliations, then rephrase them using geometric objects, by reviewing results from previous papers. The conclusions of this section are summarized in conclusion 4.3.8.

In chapter 5, this geometric description will be used to construct a functor from the ∞-category of derived foliations to an ∞-category of derived stacks equipped with extra structure, as summarized in conclusion 5.1.12. We will then show that this functor is fully faithful, and explicitly describe its essential image, which yields an equivalence of ∞-categories between derived foliations and this essential image, as stated in theorem 5.2.9

for the affine case, and generalized to relative derived Deligne-Mumford stacks in corollary 5.3.25. Note that this thesis cointains an appendix A, which consists in a collection of results on Beck-Chevalley conditions, precisely because these conditions appear in the proofs of the section 5.2 of this chapter.

In chapter 6, we will show that the push-forward functor from derived stacks over the base to derived stacks over the target preserves the extra structure of the derived stacks mentioned above, as stated in corollary 6.1.12. This will yield both the above theorem 2.2.6 and finally the main theorem 2.2.1, as well as the description of the tangent complex of the resulting derived foliations in proposition 2.2.3.

In chapter 7, we will define shifted symplectic structures on derived foliations, and prove existence theorems for these structures.

In chapter 8, we will prove two applications of the results of this thesis, mentioned in example 2.2.4 and example 2.2.5, respectively for Gromov-Witten theory and Hilbert schemes.

Finally, in chapter 9, we will talk about ongoing and future work following this thesis.

Use of generative AI: Generative AI was used for bibliographic research, for summarizing selected personal texts intended for introductory sections of other later chapters, and for correcting spelling, grammar, and typography.

CHAPTER 3

DERIVED ALGEBRAIC GEOMETRY

In this chapter, we will present derived algebraic geometry and the definitions leading up to the construction of its objects of interest. After introducing the historical context, philosophy, and applications of derived algebraic geometry in section 3.1, we will briefly recall the definitions of ∞-categories as quasi-categories in section 3.2, we will then turn to the ∞-category of homotopical rings in section 3.3 which are the affines in derived algebraic geometry, in section 3.4 we will arrive at the definitions of derived stacks and their quasi-coherent sheaves, and finally in section 3.5 we will focus on the derived stacks which are geometrically well-behaved (i.e. derived Artin and Deligne-Mumford stacks).

3.1 INTRODUCTION

Derived algebraic geometry is a modern extension of algebraic geometry in which one enlarges the class of geometric objects so as to retain the higher homological data governing intersections, quotients, and moduli problems which, in classical algebraic geometry, is usually visible only indirectly through Tor-groups, cotangent complexes, and obstruction theories. At its core lies a simple principle: many constructions in geometry are only well behaved under restrictive hypotheses such as transversality of intersections, smoothness of moduli problems, or freeness of group actions. Whenever these hypotheses fail, classical algebraic geometry often produces objects with singularities, excess intersection phenomena, or deformation spaces of the wrong dimension. Derived algebraic geometry resolves this difficulty not by excluding such situations, but by changing the ambient category of geometric objects so that the missing higher-order information, which leads to the wrong behaviour of these objects in the classical case, is instead built into the theory, so that the derived versions of those objects keep good geometric properties in the derived sense, even without those restrictive hypothesis. In a sense, derived algebraic geometry identifies objects up to deformation, and singular contexts admit deformations which resolve these singularities (e.g. non-transverse intersections can be deformed to transverse ones, and thus identified in the derived settings).

To give a more complete perspective on Derived Algebraic Geometry, and to promote some pieces of mathematical culture, we chose to place it within the broader context of the history of mathematics. Moreover, special attention was given to the history of algebraic geometry and category theory from the late nineteenth through the twentieth century.

3.1.1 Historical Context

The content of this subsection was primarily based on [KP14, Chaps. 2–13] for the history of algebra from antiquity to the nineteenth century, on [Van85, Chaps. 1–8] for developments from medieval Islamic algebra to early group theory, on [Cor04, Chaps. 1–9] for modern structural algebra, on [Gra97, pp. 1–48] and [Die85, Chaps. VII–VIII] for modern algebraic geometry, and on [Wei99, pp. 797–830] for homological methods.

Long before algebra was recognized as a distinct discipline, mathematical cultures in Egypt, Mesopotamia, the Greco-Roman world, and India were already solving problems that historians describe as algebraic, though without later symbolism or a unified concept of algebra. In Egypt, especially in the Rhind Mathematical Papyrus (c. 1650 BCE, copying older material), problems involving unknown quantities appear in contexts tied to distribution, labor, grain, and measurement [Imh16, §§ 9.1.1, 9.2–9.3, and 11.1–11.3]. In Old Babylonian Mesopotamia, verbal procedures for lengths, areas, excavations, and mixtures addressed problems that can be represented, in modern notation, by linear and quadratic equations [Høy02, Chaps. 2–5 and 7][Rob08, §§ 4.2–4.4][KP14, Chap. 2].

In later periods, such problem-solving was also shaped by astronomy, calendrics, and related predictive sciences, especially in Mesopotamia and India [Rob08, § 8.2][Plo09, §§ 4.2–4.4]. Diophantus of Alexandria (c. third century CE), in the *Arithmetica*, presented methods for solving numerical problems using natural language and some abbreviated symbols (often described as syncopated algebra) [CO13, §§ 1.2–3][KP14, Chap. 4], while Brahmagupta (598–668), in the *Brāhmasphuṭasiddhānta* of 628, stated rules for arithmetic with zero and negative quantities and procedures for solving quadratic equations [Plo09, § 5.1][KP14, Chap. 6]. Although neither work presented algebra as a separately named discipline, both treated problems that later historians have described as algebraic.

The modern term "algebra" derives from *al-jabr*, a word appearing in the title of Muḥammad ibn Mūsā al-Khwarizmi's (c. 780–c. 850) treatise *Kitāb al-jabr wa'l-muqābala*, written around 820 [Al-09]. There, algebra appears as a distinct method for reducing problems to canonical linear and quadratic equations by *restoration* (*al-jabr*) and *balancing* (*al-muqābala*). Unknown quantities are treated systematically, and their solutions are organized by standard equation types [Ras94, § I.1][Van85, Chap. 1][KP14, Chap. 7].

Later Arabic mathematicians broadened the range of problems and operations treated within algebra. Abū Kāmil Shujā$^{\text{c}}$ ibn Aslam (c. 850–c. 930) extended rhetorical algebra to more elaborate problems and irrational quantities [Kām66]; al-Karaji (c. 953–c. 1029) and al-Samaw'al (1130–1180) shifted attention toward operations on powers and polynomials [Ras94, Chap. I]; and Umar al-Khayyam (1048–1131) and Sharaf al-Din al-Tusi (c. 1135–1213) studied cubic equations, al-Khayyam through conic constructions [Kha31][Oak11, §§ 3 and 6][Van85, Chap. 1] and al-Tusi through an analysis of conditions for the existence of positive solutions [al-86][Ras94, Chap. III]. Together, these works extended algebra beyond the canonical equation types treated by al-Khwarizmi [KP14, Chap. 7, pp. 137–173].

Through translation and adaptation, these methods entered Latin Europe from the twelfth century onward. Fibonacci (c. 1170–c. 1240/1250), in the *Liber Abaci* of 1202 [Pis02], presented Hindu-Arabic numeration and algebraic methods for commercial and recreational problems, contributing to their diffusion in Latin Europe and to the later abbacco tradition. Pacioli's *Summa* of 1494 was a comprehensive printed compendium of arithmetic, algebra, practical geometry, and bookkeeping [Van85, Chap. 2][KP14, Chap. 8, pp. 178–204].

During the Renaissance, a major development was the Italian solution of cubic and quartic equations. Cardano (1501–1576), especially in the *Ars Magna* of 1545 [Car68], drawing on del Ferro, Tartaglia, and Ferrari, published methods for solving cubic and quartic equations by radicals; Bombelli (1526–1572), in *L'Algebra* of 1572, gave systematic rules for calculating with square roots of negative quantities, described today as complex numbers [KP14, Chap. 9, pp. 214–246][Tig16, §§ 2.1–2.3 and 3.1–3.2].

A further change followed with Viète (1540–1603) and Descartes (1596–1650) through their development and systematization of symbolic algebra. Viète's *In artem analyticem isagoge* of 1591 and related works [Viè83] developed a systematic literal algebra using letters for known and unknown quantities [Bos01, §§ 8.2–8.5][KP14, Chap. 9, pp. 227–246]. Descartes, in *La Géométrie* of 1637 [SL54], extended symbolic methods in his theory of equations, emphasized roots and their signs, and applied algebra systematically to geometric problems [Bos01, §§ 20.1–21.3 and 27.1–27.5][KP14, Chap. 10, pp. 261–270][Van85, Chap. 3].

In the late seventeenth and early eighteenth centuries, Newton (1642–1727) and Leibniz (1646–1716) extended the study of equations in different directions. Newton's lecture notes, printed in Latin as the *Arithmetica universalis* in 1707 and translated as *Universal Arithmetick* in 1720 [New20], treated the reduction and transformation of equations, relations between roots and coefficients, and rules concerning their roots [KP14, Chap. 10, pp. 275–287]. In a 1693 letter to l'Hospital, Leibniz described an elimination rule for systems of linear equations in a form now expressed using determinants [KP14, Chap. 12, pp. 336–339]; separately, he pursued a symbolic calculus of logic [Str69, pp. 123–129].

During the later eighteenth century and the opening years of the nineteenth, the theory of equations became more systematic. Euler's *Elements of Algebra*, first published in 1770 [Eul09], offered a broad textbook treatment of elementary algebra and equations [KP14, Chap. 10, pp. 275–287]. Lagrange's *Réflexions sur la résolution algébrique des équations* of 1770–1771 [Lag69] analyzed the known solutions of cubic and quartic equations through functions of their roots and the effects of permuting them [Tig16, §§ 9.1–10.3]. Gauss's *Disquisitiones Arithmeticae* of 1801 [Gau66] developed congruences and the arithmetic of cyclotomic equations, including the division of the circle [Tig16, §§ 12.1–12.6]. These developments connected computational algebra with increasingly structural questions about roots, permutations, and solvability.

The theory of equations was notably transformed by Abel (1802–1829) and Galois (1811–1832). Abel showed in 1824 that no general solution by radicals exists for equations of degree greater than four [Pes04, Chap. 6]. Galois, in memoirs of 1830–1832 [Neu11], related solvability by radicals to properties of groups of permutations of roots. This work helped shift the theory of equations toward structural questions about groups and solvability, although the broader transition to structural algebra occurred gradually during the nineteenth century [Tig16, Chaps. 13–14][Van85, Chaps. 4 and 6][KP14, §§ 11.2–11.5].

In the decades after Galois, groups, invariants, and ideals became increasingly prominent subjects of algebraic research. Jordan (1838–1922), in the *Traité des substitutions et des équations algébriques* of 1870 [Jor70], systematized substitution groups [Van85, Chaps. 7–8][KP14, §§ 11.5–11.6]; Cayley (1821–1895) and Sylvester (1814–1897) developed invariant theory [Cri86, pp. 241–254][Kle98, pp. 32–33][KP14, § 12.4]; and Kronecker (1823–1891) and Dedekind (1831–1916) developed distinct algebraic approaches to number theory, while Dedekind made ideals central to the treatment of divisibility in algebraic number fields [Cor04, §§ 2.2.1–2.4][Kle98, pp. 23–29]. These developments formed part of a gradual transition from the classical theory of equations to modern algebra, in which groups, invariants, and ideals became objects of study in their own right [Cor04, §§ 1.1, 3.1, and 14.1–14.2][KP14, §§ 14.1–14.2].

The passage from algebra to algebraic geometry developed in part through the study of algebraic functions and curves, algebraically defined objects with geometric interpretations. Riemann (1826–1866), in his 1857 memoir on Abelian functions [Rie57], studied algebraic curves by analytic and topological methods on what are now called Riemann surfaces. Clebsch (1833–1872), Gordan (1837–1912), Brill (1842–1935), and Max Noether (1844–1921) then developed the geometry of curves, birational transformations, and linear series. An algebraic reformulation came with Dedekind and Weber's 1882 paper *Theorie der algebraischen Functionen einer Veränderlichen* [DW82], which reframed the theory of algebraic functions using function fields and ideals [Kle98, pp. 29–31]. These developments connected the geometry of curves with the algebra of polynomial equations, function fields, and ideals [Gra97, pp. 2–6][Die85, Chap. V and §§ VI.1–VI.3].

In *Über die Theorie der algebraischen Formen* of 1890 [Hil90], Hilbert (1862–1943) proved finite-generation results including the basis theorem; in subsequent work he established the Nullstellensatz. These results became fundamental to commutative algebra and algebraic geometry [Gra97, pp. 7–11][Cor04, § 3.1]. At the same time, the Italian school and especially Segre (1863–1924), Castelnuovo (1865–1952), Enriques (1871–1946), and Severi (1879–1961) developed the birational theory of curves and surfaces, although some of its arguments were later criticized as insufficiently rigorous [Gra97, pp. 2–3][Wae71, pp. 171–180][Die85, §§ VI.4–VI.5]. Subsequent foundational work sought algebraic methods for treating multiplicities, singularities, and correspondences more rigorously.

In the early twentieth century, abstract and commutative algebra provided new foundations for algebraic geometry. Francis Sowerby Macaulay (1862–1937), in *The Algebraic Theory of Modular Systems* of 1916 [Mac16], advanced the algebra of polynomial ideals, while Emmy Noether (1882–1935), in *Idealtheorie in Ringbereichen* (1921) [Noe21], formulated ideal theory for general commutative rings satisfying finiteness conditions [Gra97, §§ 9–10][Cor04, §§ 1.3, 4.7, and 5.2–5.5]. Wolfgang Krull (1899–1971) later developed dimension theory for local rings [Kru37], providing tools subsequently used in the study of varieties and singularities. Bartel Leendert van der Waerden (1903–1996), through *Moderne Algebra* (1930–1931) and his papers in algebraic geometry, helped transmit these structural methods into geometry. Oscar Zariski (1899–1986), who had been trained in the Italian tradition, reformulated parts of the subject using ideals, local rings, valuations, and the study of singularities, notably in *Algebraic Surfaces* (1935) [Zar35] and later work on local uniformization. André Weil (1906–1998) then presented systematic foundations in *Foundations of Algebraic Geometry* (1946) [Wei46], where varieties over arbitrary fields, intersection multiplicities, and abstract methods received a systematic treatment [Die85, Chap. VII, §§ 2–7][Wae71, pp. 171–180]. By the mid-1940s, algebraic geometry had become closely integrated with modern algebra.

The further development of algebraic geometry after Weil incorporated sheaf theory, cohomology, and homological algebra, whose early formulations arose largely in topology and complex analysis. Jean Leray (1906–1998), in wartime papers beginning with *Structure de l'anneau d'homologie d'une représentation* (1946) [Ler46], developed sheaf-theoretic and spectral-sequence methods that Henri Cartan (1904–2008) and his collaborators subsequently reformulated and extended in the Séminaire Cartan of 1950–1951 [Car51a; Car51b][Cho10, pp. 52–73][Wei99, pp. 811–816]. At almost the same moment Samuel Eilenberg (1913–1998) and Saunders Mac Lane (1909–2005), working from problems in algebraic topology, introduced categories, functors, and natural transformations in *General Theory of Natural Equivalences* (1945) [EM45][Krö07, §§ 3.1–3.3 and 4.1–4.3], and Cartan with Eilenberg codified homological algebra in *Homological Algebra* (1956) [CE56]. Jean-Pierre Serre (1926–), through *Homologie singulière des espaces fibrés. I: La suite spectrale* (1950) [Ser50], applied spectral sequences to fiber spaces; through *Faisceaux algébriques cohérents* (1955) [Ser55] and *Géométrie algébrique et géométrie analytique* (1956) [Ser56], he then brought sheaf cohomology and comparison methods into algebraic geometry. These developments increasingly organized the study of geometric objects through sheaves, cohomological invariants, and categorical constructions.

Alexander Grothendieck (1928–2014) carried this transformation further by generalizing both the geometric objects and the categorical language used to study them. In *Sur quelques points d'algèbre homologique* (1957) [Gro57] he developed a general theory of abelian categories and derived functors, with applications to sheaf cohomology [Cor04, § 8.4]; with Jean Dieudonné (1906–1992) he then built scheme theory in *Éléments de géométrie algébrique* from 1960 onward [EGA]. His Bourbaki talks, collected in *Fondements* *de la géométrie algébrique* (1962) [Gro62], developed representability methods and the study of families and moduli [Die85, Chap. VIII, §§ 1–3][McL16, pp. 256–265]. Jean-Louis Verdier (1935–1989), working in the same circle, developed the formal theory of derived categories, conceived in the 1960s and published posthumously as *Des catégories dérivées des catégories abéliennes* (1996) [Ver96], providing a framework in which complexes and quasi-isomorphisms could be treated systematically [Wei99, pp. 822–826]. By this stage, algebraic geometry was routinely formulated using commutative algebra, categorical organization, and cohomological methods.

These tools were soon applied to moduli and deformation problems that required geometric objects and infinitesimal methods beyond ordinary schemes. Pierre Deligne (1944–) and David Mumford (1937–), in *The irreducibility of the space of curves of given genus* (1969) [DM69], used algebraic stacks and stable curves to construct a compactification and prove the irreducibility of the moduli space of smooth curves; Michael Artin (1934–), in *Versal deformations and algebraic stacks* (1974) [Art74], related versal deformation theory to algebraic stacks; and Luc Illusie (1940–), in *Complexe cotangent et déformations I* (1971) [Ill71], supplied the cotangent-complex formalism for infinitesimal deformation theory. Stacks retain automorphism data in moduli problems, while cotangent complexes organize infinitesimal deformations and obstructions. These constructions supplied part of the stack-theoretic and deformation-theoretic background later used in derived algebraic geometry [Toë14a, § 1, esp. pp. 160–164 and 166–170].

In parallel, algebraic topology and category theory developed formal devices that later proved important for higher and derived geometry. Category theory was developed further in directions relevant to geometry by Grothendieck's theory of topoi [McL90, §§ 2–5] and by F. William Lawvere (1937–2023), whose joint work with Myles Tierney in *Quantifiers and Sheaves* (1970) [Law71] gave an elementary axiomatization of topoi extending Grothendieck's earlier sheaf-theoretic framework [McL90, § 5, pp. 359–363]. Daniel Quillen (1940–2011), in *Homotopical Algebra* (1967) [Qui67], introduced model categories and a general calculus of derived functors for homotopy-theoretic settings [Wei99, pp. 822–826]. J. Michael Boardman (1938–2021) and Rainer Vogt (1942–2015), in *Homotopy Invariant Algebraic Structures on Topological Spaces* (1973) [BV73], developed methods for homotopy-invariant algebraic structures; J. F. Jardine (1951–), in *Simplicial presheaves* (1987) [Jar87], constructed a local model structure on simplicial presheaves. Together, model categories, homotopy-coherent algebra, and simplicial presheaves supplied tools later used in theories of higher stacks and derived geometric objects [Toë14a, § 1, esp. pp. 166–170].

These branches converged in the development of higher stacks and derived algebraic geometry. Carlos Simpson (1962–), in *Algebraic (geometric) n-stacks* (1996) [Sim96], developed a theory of algebraic n-stacks using simplicial presheaves, while André Joyal (1943–), in *Quasi-categories and Kan complexes* (2002) [Joy02], studied quasi-categories and their relation to Kan complexes. Bertrand Toën (1973–) and Gabriele Vezzosi (1966–), in *Homotopical Algebraic Geometry I* (2005) [HAGI] and *II* (2008) [HAGII], developed homotopical algebraic geometry over symmetric monoidal model categories and applied this theory to define and study derived algebraic geometries; and Jacob Lurie (1977–), in *Derived Algebraic Geometry* (2004) [LuriePhD] and subsequent papers, also developed a foundation for derived schemes and stacks based on simplicial commutative rings and higher-categorical methods, to which he later gave extensive foundations in *Higher Topos Theory* (2009) [HTT] and *Higher Algebra* (2017) [HA]. Derived algebraic geometry thereby extends schemes and stacks by allowing derived rings and homotopical structure, retaining higher information in intersections, deformations, and moduli problems [Toë14a, Introduction and § 1].

3.1.2 Philosophy

Derived algebraic geometry can be viewed as a double enlargement of classical algebraic geometry. The first enlargement is the passage from schemes to stacks. The second is the passage from ordinary commutative rings to homotopical commutative rings, such as simplicial commutative rings or, in characteristic zero, commutative differential graded algebras in nonpositive degrees. These two directions repair different deficiencies of classical moduli theory. Stack theory corrects the *target* of the functor of points: many moduli problems are not naturally set-valued, because objects have automorphisms and families are classified only up to isomorphism. Derived geometry corrects the *source* of the functor of points: ordinary rings do not record enough infinitesimal or homological information, so deformation and intersection problems lose the higher data carried by Tor-groups, Ext-groups, and the cotangent complex. This distinction is explicit in the foundational texts. Lurie stresses that algebraic stacks are introduced to make moduli functors representable, whereas derived geometry should be compared not to stacks but to the passage from reduced to nonreduced schemes, namely an enlargement of the class of test algebras that detects more geometry [LuriePhD]. Toën and Vezzosi express the same idea in the broader framework of geometric stacks and homotopical algebraic geometry [HAGII; Toë14a].

The side of stacks begins with the functor-of-points philosophy. A scheme can be reconstructed from the functor it defines on commutative rings, together with descent. But moduli functors rarely behave like schemes or even algebraic spaces. The problem is not only that quotients may be singular or nonseparated, but more fundamentally that the relation identifying two objects of a moduli problem is itself geometric: curves, sheaves, vector bundles, and principal bundles come with automorphism groups, and those groups vary in families. If one insists on set-valued functors, one forgets this isotropy and usually destroys descent. The correct target is therefore groupoids, and more generally spaces or ∞-groupoids. In that language, a stack is a moduli functor with descent whose values remember objects, isomorphisms, and higher compatibilities. The foundational paper

[HAGII] formulates this systematically: geometric stacks arise by combining the functorial viewpoint with homotopy-theoretic descent, so that quotients and moduli problems can be represented in a category larger than schemes but still recognizably algebro-geometric. Many important derived moduli problems are not represented by derived schemes but by derived Artin stacks [Toë14a].

Moreover, on stacks, the deformation-theoretic motivation is closely tied to the cotangent complex. Classical moduli spaces are often singular because deformations are obstructed. The right linear object attached to a moduli problem is then not a vector bundle but a complex. Quillen, Grothendieck, and Illusie showed that the cotangent complex governs derivations, first-order deformations, and obstructions. Derived algebraic geometry includes this data into the structure of its objects. A derived stack is not merely a stack together with a chosen obstruction theory, its cotangent complex is intrinsic, and obstruction theory is built into the geometry. Toën emphasizes that derived Artin stacks possess cotangent complexes and that these control obstruction theory in a functorial global form [Toë14a]. In [HAGII], this structural statement is made precise: geometric stacks in the homotopical setting satisfy infinitesimal cartesian conditions and therefore come with obstruction theories by construction. As such, this theory replaces many ad hoc obstruction calculations by geometric objects.

The derived motivation, as presented by Lurie, enters in two stages. First, passing from reduced schemes to all schemes enlarges the class of test rings and reveals other features of objects, for instance tangent directions appear by evaluation of functors of points on dual numbers. Second, passing from ordinary schemes to derived schemes enlarges the class of test rings again, now to homotopical rings, and reveals higher deformation classes and obstruction classes [LuriePhD]. In his introductory example, $H^1(X, T_X)$ classifies first-order deformations of a smooth projective variety X, while $H^2(X, T_X)$ classifies a nonclassical deformation over a derived infinitesimal base. In this way an obstruction group becomes a space of points of the same moduli problem, evaluated on a richer affine test object. The obstruction is no longer an external defect in the deformation theory but made geometric as a derived infinitesimal direction. This is one of the decisive philosophical shifts of the subject.

The same idea also explains the role of homotopical affine schemes in intersection theory. Classical scheme-theoretic intersection already improves set-theoretic intersection because nilpotents measure first-order failure of transversality, but it is still incomplete in singular or nontransverse situations, where the correct intersection multiplicity is governed by Serre's formula and hence by the full complex $\mathcal{O}_Y \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_Z$. Lurie uses this as the opening example of derived algebraic geometry. The ordinary tensor product remembers only the zeroth term, while the derived tensor product remembers all higher Tor-groups, hence the full excess-intersection information. More concretely, if $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} A$, and $Z = \operatorname{Spec} B$, then the derived fiber product

$$Y \times^{\mathbf{R}}_{X} Z$$

is the spectrum of the affine ring $A \otimes^{\mathbf{L}}_{R} B$, whose homotopy groups satisfy

$$\pi_i\Big(A \otimes^{\mathbf{L}}_{R} B\Big) \simeq \operatorname{Tor}^{R}_{i}(A, B).$$

Toën formulates the same point by saying that derived algebraic geometry is designed for "bad intersections" and other nongeneric situations, in contrast with classical geometry, which is most transparent in transverse situations [Toë14a]. When the intersection is transverse, the higher homotopy sheaves vanish and the derived intersection collapses to the classical one. When transversality fails, the higher homotopy sheaves record the missing intersection data.

Finally, another core idea of the subject is to use derived geometry as presented above to gain information on classical geometry. A derived scheme or derived stack X comes with a truncation $t_0(X)$, obtained by discarding higher homotopy. The morphism $t_0(X) \to X$ behaves like a formal thickening, and Toën emphasizes that the difference between classical and derived geometry is concentrated at this formal level [Toë14a]. The higher structure sheaves $\pi_i(\mathcal{O}_X)$ are quasi-coherent modules on $t_0(X)$, which encode successive infinitesimal corrections to the underived moduli problem. The classical space remains visible as $t_0(X)$, but it is supplied with canonical higher data coming from X that explains singularities, obstructions, and excess intersections. In particular, since homotopy limits and colimits are better behaved than classical ones, derived intersections and quotient constructions acquire the functorial behavior that classical formulas predict only under restrictive hypotheses, and their derived structures induce extra classical structure on their truncations in a canonical way.

In conclusion, the interaction between the stacky and derived directions can be summarized by saying that stackiness remembers automorphisms and descent, while derivedness remembers higher infinitesimal and homological data. The moduli problems that motivate algebraic geometry usually require both : for example, a moduli stack of sheaves or complexes is naturally stacky because objects have automorphisms, and it is naturally derived because its deformation theory is controlled by a complex whose higher cohomology groups give obstructions. At the foundational level, the theory enables the cotangent complexes, tangent complexes, and derived intersections to become intrinsic constructions. At the pratical level, it enables classical truncations of derived moduli stacks to inherit canonical obstruction theories and virtual structures [Toë14a; STV15]. In that sense, derived algebraic geometry is a correction principle: many classical moduli spaces become geometrically natural only after one remembers both their stacky symmetries and their derived infinitesimal directions.

3.1.3 Applications

The applications of derived algebraic geometry follow its philosophy. As explained above, one replaces a classical moduli space by a derived moduli stack whose extra homotopical structure records obstructions, excess intersections, and automorphisms. One then uses that richer geometry to construct invariants or structures that, in the underived world, are difficult to define, unnatural in the functorial sense, or incomplete. Three application areas dominate the literature: virtual intersection theory and enumerative geometry, shifted symplectic geometry on moduli spaces, and quantization [Toë14a; PTVV].

i) **Enumerative geometry.** Classical enumerative problems rarely produce smooth moduli spaces of the expected dimension. Stable maps, sheaves on Calabi-Yau varieties, and related moduli spaces are often singular and have components of the wrong dimension. The standard underived remedy is a perfect obstruction theory, which yields a virtual tangent bundle and a virtual fundamental class. In derived algebraic geometry, a quasi-smooth derived stack already contains the obstruction theory in its cotangent complex, so the virtual class is not extra input but an intrinsic shadow of the derived enhancement. Toën explains this at the level of general principle: the truncation $t_0(X)$ of a derived scheme inherits a perfect obstruction theory from the morphism $t_0(X) \to X$, and this suffices to define virtual classes, although it forgets finer derived information [Toë14a]. Schürg, Toën, and Vezzosi make this concrete for moduli of maps and perfect complexes. A quasi-smooth derived enhancement yields a functorial perfect obstruction theory in the sense of Behrend and Fantechi, and this recovers the standard and reduced obstruction theories that appear in Gromov-Witten theory and sheaf-counting problems [STV15].

A more systematic version is due to Khan, who constructs virtual fundamental classes of quasi-smooth derived Artin stacks in a bivariant formalism [Kha19]. In that framework, excess intersection, nontransverse Bézout formulas, and Grothendieck-Riemann-Roch statements become natural consequences of derived geometry rather than separate technical results [Kha19]. While the virtual class is often described as a substitute for the fundamental class of a singular moduli space, derived algebraic geometry suggests that it is the fundamental class of the derived moduli stack, transported to its classical truncation. Once phrased this way, many compatibilities are reinterpreted as ordinary functoriality for derived pullbacks and quasi-smooth morphisms. Moreover, the application to Gromov-Witten theory has also been studied. Mann and Robalo explain how derived algebraic geometry enriches the structures of the moduli of stable maps, makes the hidden operadic structure more geometric, and gives a conceptual account of the virtual objects used in Gromov-Witten theory [MR21]. In this setting, the derived enhancement encodes exactly the excess deformation data that the virtual cycle is meant to recover.

ii) **Shifted Symplectic Geometry.** One of the major achievements of the subject is the observation that many moduli stacks which are singular or stacky in the ordinary sense nevertheless carry canonical symplectic structures once they are viewed as derived

Artin stacks. Pantev, Toën, Vaquié, and Vezzosi define n-shifted symplectic structures and prove several existence theorems [PTVV]. Two base examples already show the strength of the theory. The classifying stack BG of a reductive group carries canonical 2-shifted symplectic forms attached to invariant quadratic forms on its Lie algebra, and the derived stack $\mathbb{R}\mathrm{Perf}$ of perfect complexes carries a canonical 2-shifted symplectic form defined by the trace pairing. The mapping-stack theorem is even more powerful. If F is n-shifted symplectic and X is an oriented object of dimension d, then $\mathrm{Map}(X, F)$ is $(n-d)$-shifted symplectic. This single theorem recovers many classical symplectic structures and extends them to singular and higher-stacky settings.

Moreover, concrete geometric consequences of these theorems are also given. If X is a smooth proper Calabi-Yau variety of dimension d, then the derived moduli stack of perfect complexes on X carries a $(2-d)$-shifted symplectic structure. If M is a compact oriented topological manifold, then the derived moduli stack of G-local systems on M is $(2-\dim M)$-shifted symplectic. In dimension two these become ordinary 0-shifted symplectic structures, so the smooth loci recover familiar symplectic moduli spaces of sheaves, flat bundles, and representations. The derived statement, beyond recovering these results, explains the existence of these forms, controls their behavior under singularities, and gives a clean intersection principle: if $Y \to X$ and $Z \to X$ are Lagrangians in an n-shifted symplectic derived stack, then the derived fiber product $Y \times_X^{\mathbf{R}} Z$ is $(n-1)$-shifted symplectic. This principle underlies derived critical loci, singular symplectic reduction, and parts of Donaldson-Thomas theory.

iii) **Quantization.** Shifted Poisson and shifted symplectic structures provide the input for deformation quantization on moduli stacks that are typically singular, stacky, or both. In the derived setting, the quantization problem is formulated not for the algebra of functions on a smooth space, but for the dg-category of sheaves on a derived stack. More precisely, an n-shifted Poisson structure on a derived Artin stack X is expected to produce a formal deformation of $L_{\mathrm{qcoh}}(X)$, viewed as an E_n-monoidal dg-category; this is the perspective developed by Toën and carried out in the work of Calaque, Pantev, Toën, Vaquié, and Vezzosi on shifted Poisson structures and deformation quantization [Toë14b; CPTVV]. In many examples coming from moduli of bundles, local systems, or complexes, the relevant shifted symplectic structures are produced by the mapping-stack formalism of [PTVV], whose relation with the AKSZ construction of [ASZK97] was clarified by Calaque and lies behind the field-theoretic origin of several quantization procedures [Cal15]. The result is a uniform picture in which quantum groups, skein-type structures, and quantizations of moduli spaces appear as instances of a single derived mechanism.

Geometric quantization admits a parallel shifted version. Safronov introduces geometric quantization for shifted symplectic stacks by replacing the classical ingredients of prequantum line bundles and polarizations with derived analogues, namely prequantizations and Lagrangian fibrations [Saf23]. This framework is designed to produce categories or linear objects attached to shifted symplectic moduli problems in a way compatible with their derived and stacky nature, rather than only with a smooth symplectic locus. It applies to examples such as symplectic groupoids, Hamiltonian spaces, and moduli stacks of flat connections, and it proves in the Hamiltonian setting a derived form of the principle that quantization commutes with reduction.

These applications give concrete illustrations of the interest of derived algebraic geometry in several active questions in geometry, and show how its conceptual principles enable the existence and good behaviour of relevant objects in practice.

3.2 Infinity-Categories

After having introduced the context and ideas of derived algebraic geometry in the previous section, this section and the next ones will be devoted to formally introduce definitions and properties that lead to the construction of its objects. As mentioned in the opening paragraph of this chapter, we will introduce ∞-category theory in this section, and then the ∞-categories of derived affines, derived stacks, and finally geometric stacks in the next three sections. Since this section and the next ones do not aim to provide a complete set of formal definitions, so as to remain concise, we will deliberately avoid introducing some of them. Instead, we will always point the reader to the corresponding exact definition number in a complete reference, which will mainly be [HTT] in this section. Note that by ∞-*category* we mean $(\infty, 1)$-*category*, as is standard in the field. Finally, as is standard in category theory, due to size issues, one should normally make precise the distinction between small categories (where collections of objects and/or morphisms are sets) and big categories (where they form a proper class), or use a machinery of Grothendieck universes or large cardinals to tract size issues. However, this is beyond the scope of this section, and this discussion will thus be omitted.

3.2.1 Quasi-Categories

In this section, we will use quasi-categories as our preferred model of ∞-categories, following [HTT, Def. 1.1.2.4]. They are simplicial sets satisfying a condition known as the *horn-filling condition*, introduced by André Joyal in [Joy02]. As such, we will start by very briefly recalling the definition of simplicial sets following [GJ99].

Definition 3.2.1. *Let Δ denote the simplex category, whose objects are the finite nonempty linearly ordered sets*

$$[n] = \{0 < 1 < \cdots < n\}, \qquad n \geq 0.$$

A simplicial set *is a contravariant functor*

$$X \colon \Delta^{\mathrm{op}} \to \mathrm{Set}.$$

Equivalently, it is a collection of sets $(X_n)_{n\geq 0}$*, where elements of* X_n *are called* n-simplices *of* X*, together with face maps* $d_i : X_n \to X_{n-1}$ *and degeneracy maps* $s_j : X_n \to X_{n+1}$ *satisfying simplicial identities [GJ99, I.1, eq. (1.3)]. The category of simplicial sets is denoted by* sSet*.*

Definition 3.2.2. *For* $n \geq 0$*, the* standard n-simplex *is the representable simplicial set*

$$\Delta^n := \mathrm{Hom}_\Delta(-, [n]).$$

Its i*th face is the image of the coface map* $d^i \colon [n-1] \to [n]$*, and the* boundary $\partial\Delta^n \subseteq \Delta^n$ *is the simplicial subset generated by all faces of the nondegenerate simplex* $\iota_n \in (\Delta^n)_n$ *[GJ99, I.1, Ex. 1.7].*

Definition 3.2.3. *For* $0 \leq i \leq n$ *and* $n \geq 1$*, the* ith horn $\Lambda^n_i \subseteq \Delta^n$ *is the simplicial subset generated by all faces of* Δ^n *except the* i*th face. It is called* inner *if* $0 < i < n$ *[GJ99, I.1, Ex. 1.7].*

Definition 3.2.4. *For an ordinary category* $\mathcal{C}$*, its* nerve $N(\mathcal{C})$ *is the simplicial set defined by*

$$N(\mathcal{C})_n = \mathrm{Fun}([n], \mathcal{C}),$$

so that an n*-simplex of* $N(\mathcal{C})$ *is a composable string*

$$C_0 \to C_1 \to \cdots \to C_n$$

of n *morphisms in* $\mathcal{C}$ *[GJ99, I.1, Ex. 1.4].*

Definition 3.2.5. *A simplicial set* K *is a* Kan complex *if every horn inclusion* $\Lambda^n_i \hookrightarrow \Delta^n$ *admits a filler [HTT, Def. 1.1.2.1]. Note that the term "Kan complex" originally appeared in [BV73], referencing the original definition in [Kan57]. The category of Kan complexes will be denoted by* Kan*. In a sense we will not make more precise here, it is a model of the homotopy theory of spaces, so Kan complexes should be thought of as homotopy types of spaces, there is a notion of weak equivalence of Kan complexes, and a Kan complex* K *is said to be contractible if the canonical map* $K \to \bullet = \Delta^0$ *is a weak equivalence. See [GJ99, Chap. I].*

Definition 3.2.6. *A* quasi-category *(called an* ∞-category *in [HTT]) is a simplicial set* C *such that for every* $n \geq 2$ *and every inner horn inclusion* $\Lambda^n_i \hookrightarrow \Delta^n$ *with* $0 < i < n$*, every map*

$$\Lambda^n_i \to C$$

extends to a map $\Delta^n \to C$ *[HTT, Def. 1.1.2.4]. Elements of the set* C_0 *will be called* objects *of* C*, and by abuse of notation* $x \in C$ *means that* $x \in C_0$ *is an object. Similarly,* maps *are elements of* C_1*, whose source and target objects are provided by the face maps of the simplicial structure of* C*, and elements of* C_n *for* $n \geq 2$ *are* higher homotopies *within* C*. The category of quasi-categories will be denoted by* qCat*.*

Remark 3.2.7. The quasi-category condition is weaker than the Kan condition: one requires fillers only for inner horns, not for all horns [HTT, Def. 1.1.2.1 and Def. 1.1.2.4].

Example 3.2.8. Every Kan complex is a quasi-category, and the nerve of every ordinary category is a quasi-category [HTT, Ex. 1.1.2.5 and Ex. 1.1.2.6].

Proposition 3.2.9. *A simplicial set K is isomorphic to the nerve of an ordinary category if and only if every inner horn in K admits a unique filler [HTT, Prop. 1.1.2.2], and in particular is a quasi-category. Moreover, the nerve functor N is fully faithful, so 1-categories can be identified with the quasi-categories admitting unique horn-fillings.*

Definition 3.2.10. *If C and D are simplicial sets, let* $\mathrm{Fun}(C, D)$ *denote the internal Hom simplicial set of simplicial maps from C to D. If D is a quasi-category, then* $\mathrm{Fun}(C, D)$ *is again a quasi-category; in particular, if C and D are quasi-categories,* $\mathrm{Fun}(C, D)$ *is the* quasi-category of functors *from C to D [HTT, §1.2.7, esp. Prop. 1.2.7.3(1)]. For $F, G : C \to D$ elements of* $\mathrm{Fun}(C, D)_0$*, a map $\eta : F \implies G$ in* $\mathrm{Fun}(C, D)_1$ *is called a* natural transformation.

Remark 3.2.11. Using the above definitions, we can see that 1-category theory embeds in ∞-category theory. We will not cite all the statements that would be needed to give formal content to this statement, but it should be noted for instance that if C and D are 1-categories, the above construction recovers the nerve of the usual functor category $\mathrm{Fun}(C, D)$. More generally, constructions and concepts in ∞-category theory that are analogous to constructions or concepts coming from 1-category theory (e.g. limits, colimits, functors, adjunctions, etc) coincide with the latter when applied to ∞-categories that are in fact the nerves of 1-categories. From now on, we will identify 1-categories with their nerves, and view them as quasi-categories.

Definition 3.2.12. *The inclusion of 1-categories in quasi-categories admits a left adjoint h. For C a quasi-category, the 1-category hC will be called the* homotopy category of C *[HTT, Prop. 1.2.3.1]. The co-unit of this adjunction yields a canonical map $[-] : C \to hC$.*

Definition 3.2.13. *Given a map $f : x \to y$ in an quasi-category C, f is said to be an equivalence if $[f]$ is an isomorphism in hC, i.e. if there exists a map $g : y \to x$ together with homotopies $g \circ f \simeq Id_x$ and $f \circ g \simeq Id_y$ [HTT, Prop. 1.2.4.1].*

Definition 3.2.14. *There is a quasi-category* qCat *of quasi-categories (up to size issues), whose objects are quasi-categories, morphisms are functors, and homotopies are natural equivalences of functors. For $F : C \to D$ a functor of quasi-categories, it is an* equivalence of quasi-categories *if there exists a functor $G : D \to C$ with natural equivalences $G \circ F \simeq Id_C$ and $F \circ G \simeq Id_D$.*

Proposition 3.2.15. *Let C be a quasi-category. Then C is a Kan complex if and only if hC is a groupoid, or equivalently if all maps in C are equivalences [HTT, Prop. 1.2.5.1].*

Remark 3.2.16. If C is a quasi-category, the largest simplicial subset $C^{\simeq} \subseteq C$ spanned by equivalences is a Kan complex [HTT, Prop. 1.2.5.3].

Definition 3.2.17. *Let C be a quasi-category and let $x, y \in C_0$ be objects of C. The* mapping space *from x to y is the pull-back*

$$\begin{tikzcd}
\mathbf{Map}_C(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, C) \arrow[d, "(s,t)"] \\
\bullet \arrow[r, "(x,y)"'] & C \times C
\end{tikzcd}$$

where the bottom map map picks the object (x, y) in $C \times C$, while the right-map takes a functor $F : \Delta^1 \to C$ and evaluates it at 0 for s and 1 for t. Another concrete model is the simplicial set $\mathrm{Hom}^R_C(x, y)$ of right morphisms, defined in [HTT, §1.2.2].

Proposition 3.2.18. *In the above situation, $\mathbf{Map}_C(x, y)$ is a quasi-category since they are stable by limits, and moreover its homotopy-category $h\mathbf{Map}_C(x, y)$ is a groupoid, so it is in fact a Kan complex by the previous proposition.*

Remark 3.2.19. The above proposition can be interpreted as saying that, contrary to the case of 1-categories where $\mathrm{Hom}(x, y)$ is simply a set, for quasi-categories $\mathbf{Map}(x, y)$ is a space. Moreover, this space allows us to give a more conceptual description of the homotopy-category hC of a quasi-category C.

Definition 3.2.20. *The inclusion of sets (viewed as discrete 1-categories) in quasi-categories also admits a left adjoint π_0, and for a quasi-category C, the set $\pi_0(C)$ will be called the* connected components *of C. Note that connected components are most often applied to Kan complexes, where it computes the usual connected components of spaces in homotopy theory [GJ99, Chap. I, §7].*

Proposition 3.2.21. *If C is a quasi-category, the homotopy category hC has the same objects as C, and for $x, y \in C$ such objects there is a canonical natural bijection*

$$\mathrm{Hom}_{hC}(x, y) \cong \pi_0 \mathbf{Map}_C(x, y)$$

[HTT, Prop. 1.2.3.9].

Remark 3.2.22. Using the notion of the mapping space $\mathbf{Map}_C(x, y)$, we can recover other usual notions of 1-category theory. We will now present initial and final objects, limits and colimits (which correspond to homotopy limits and homotopy colimits in model category theory), the Yoneda embedding, and adjunctions.

Definition 3.2.23. *Let C be a quasi-category. An object $x \in C$ is* final *if it is final in hC [HTT, Def. 1.2.12.1]; for quasi-categories this is equivalent to being* strongly final *in the sense of [HTT, Def. 1.2.12.3 and Cor. 1.2.12.5]. Dually one defines* initial *objects [HTT, Rem. 1.2.12.6].*

Proposition 3.2.24. *Let C be a quasi-category and $x \in C$. Then x is final if and only if $\mathbf{Map}_C(y, x)$ is a contractible space for every $y \in C$; dually, x is initial if and only if $\mathbf{Map}_C(x, y)$ is contractible for every $y \in C$ [HTT, Prop. 1.2.12.4 and Cor. 1.2.12.5].*

Definition 3.2.25. *Let $p\colon K \to C$ be a diagram in a quasi-category C, where K is any simplicial set. The* overcategory *$C_{/p}$ is the quasi-category characterized by the universal property of [HTT, Prop. 1.2.9.2]; the* undercategory *$C_{p/}$ is its dual, as in [HTT, Rem. 1.2.9.5]. Informally, $C_{/p}$ (resp. $C_{p/}$) is the quasi-category of cones (resp. co-cones) over p, so objects of $C_{/p}$ (resp. $C_{p/}$) are diagrams $K^{\triangleleft} \to C$ (resp. $K^{\triangleright} \to C$) whose restriction to K are p, where $K^{\triangleleft}$ (resp. $K^{\triangleright}$) denotes the simplicial set K with an initial (resp. final) object formally added.*

Definition 3.2.26. *A* colimit *of $p\colon K \to C$ is an initial object of $C_{p/}$, and a* limit *of p is a final object of $C_{/p}$ [HTT, Def. 1.2.13.4].*

Definition 3.2.27. *The* ∞-category of spaces *is the quasi-category of Kan complexes, denoted by $\mathcal{S}$, in which equivalences coincide with weak equivalence of Kan complexes [HTT, Def. 1.2.16.1]. Although related, it should not be confused with the nerve of the 1-category* Kan*.*

Definition 3.2.28. *For a simplicial set K, the* presheaf ∞-category *on K is*

$$\mathcal{P}(K) := \mathrm{Fun}(K^{\mathrm{op}}, \mathcal{S})$$

[HTT, Def. 5.1.0.1].

Definition 3.2.29. *For a small quasi-category C, the* Yoneda embedding *is the functor*

$$j\colon C \to \mathcal{P}(C) = \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S}), \qquad x \mapsto \mathbf{Map}_C(-, x),$$

[HTT, Def. 5.1.0.1 and Prop. 5.1.3.1].

Proposition 3.2.30. *The Yoneda embedding*

$$j\colon C \to \mathcal{P}(C)$$

is fully faithful [HTT, Prop. 5.1.3.1]. In particular, in a quasi-category C, the data of an equivalence between two objects x and y is equivalent to the data of a natural equivalence $\mathbf{Map}_C(a, x) \simeq \mathbf{Map}_C(a, y)$ for all objects $a \in C$. Moreover, if $\mathcal{D}$ is a quasi-category which admits small colimits, then composition with j induces an equivalence

$$\mathrm{Fun}^L(\mathcal{P}(C), \mathcal{D}) \simeq \mathrm{Fun}(C, \mathcal{D})$$

where Fun^L *denote the quasi-category of functors that commute with colimits [HTT, Thm. 5.1.5.6].*

Definition 3.2.31. *Let C and D be quasi-categories. For functors $F\colon C \to D$ and $G\colon D \to C$, an* adjunction $F \dashv G$ *is the data of a* unit transformation

$$\eta\colon Id_C \to G \circ F$$

such that for every $x \in C$ and $y \in D$, the induced map

$$\mathbf{Map}_D(Fx, y) \to \mathbf{Map}_C(GFx, Gy) \xrightarrow{\eta(x)^*} \mathbf{Map}_C(x, Gy)$$

is an equivalence [HTT, Def. 5.2.2.7 and Prop. 5.2.2.8]. As in 1*-category theory, this implies and is equivalent to the data of a* co-unit transformation

$$\epsilon\colon F \circ G \to Id_D$$

such that for every $x \in C$ and $y \in D$, the induced map

$$\mathbf{Map}_C(x, Gy) \to \mathbf{Map}_D(Fx, FGy) \xrightarrow{\epsilon(y)_*} \mathbf{Map}_D(Fx, y)$$

is an equivalence. Moreover, η and ϵ satisfy the usual triangle identities up to higher homotopy coherences. Finally, F and G are equivalences of quasi-categories inverse to each other if and only if η and ϵ are natural equivalences.

Proposition 3.2.32. *Let*

$$F\colon C \rightleftarrows D\colon G$$

be an adjoint pair of quasi-categories, with $F \dashv G$. Then F preserves all colimits that exist in C, and G preserves all limits that exist in D. Equivalently, if

$$\overline{p}\colon K^{\triangleright} \to C$$

is a colimit diagram, then

$$F \circ \overline{p}\colon K^{\triangleright} \to D$$

is again a colimit diagram; dually, if

$$\overline{q}\colon K^{\triangleleft} \to D$$

is a limit diagram, then

$$G \circ \overline{q}\colon K^{\triangleleft} \to C$$

is again a limit diagram [HTT, Prop. 5.2.3.5].

Remark 3.2.33. The previous proposition shows that adjoint functors automatically preserve the corresponding universal constructions: left adjoints preserve colimits, and right adjoints preserve limits. The next results explain a converse under presentability hypotheses: for presentable quasi-categories, a functor which preserves small colimits admits a right adjoint, while a functor which is accessible and preserves small limits admits a left adjoint [HTT, Cor. 5.5.2.9]. This is an ∞-categorical analogue of adjoint functors theorems.

Definition 3.2.34 (Presentable quasi-categories)**.** *A quasi-category C is said to be* presentable *if it admits small colimits and there exists a regular cardinal κ such that:*

1. *C is locally small and admits κ-filtered colimits;*
2. *the full subcategory $C^{\kappa} \subseteq C$ spanned by the κ-compact objects is essentially small, where an object $c \in C$ is* κ-compact *if for every small κ-filtered diagram $p\colon K \to C$, with K a simplicial set having fewer than κ nondegenerate simplices, the natural map*

   $$\operatorname*{colim}_{k \in K} \mathbf{Map}_C(c, p(k)) \longrightarrow \mathbf{Map}_C\Big(c, \operatorname*{colim}_K p\Big)$$

   is an equivalence [HTT, Def. 5.3.4.5];
3. *C^{κ} generates C under small κ-filtered colimits.*

Equivalently, C is presentable if it is accessible and admits small colimits [HTT, Prop. 5.4.2.2, Def. 5.5.0.1].

Proposition 3.2.35 (Adjoint functor theorem for presentable quasi-categories)**.** *Let*

$$F\colon C \to D$$

be a functor between presentable quasi-categories.

1. *If F preserves small colimits, then F admits a right adjoint.*
2. *If F is accessible and preserves small limits, then F admits a left adjoint.*

Equivalently:

- *F is a left adjoint if and only if it preserves small colimits;*
- *F is a right adjoint if and only if it is accessible and preserves small limits.*

This is the adjoint functor theorem for presentable ∞-categories [HTT, Cor. 5.5.2.9].

Definition 3.2.36. *The sub-quasi-category of* qCat *spanned by presentable ∞-categories as objects and left adjoints as morphisms is denoted by* Pr^{L}*.*

Remark 3.2.37. While other important constructions and properties of ∞-category theory exist and are developped throughout [HTT], we will stop our exposition here since the above are enough for the purpose of this section. From now on, we will always use the denomination ∞-category, and avoid explicit reference to the model of quasi-categories.

3.2.2 Infinity-Toposes

We will now present two important classes of ∞-category, ∞-topoi in this subsection, and stable ∞-categories in the next one, which correspond intuitively to ∞-categories of spaces and quantities respectively (see [Law92] for more details on this analogy in category theory).

Definition 3.2.38 (Sieves and Grothendieck topologies)**.** *Let C be a small ∞-category. A* sieve *on an object $c \in C$ is a sieve on the overcategory $C_{/c}$, in the sense of [HTT, Def. 6.2.2.1]. A* Grothendieck topology *on C is the datum, for every object $c \in C$, of a collection of sieves on c, called* covering sieves*, satisfying the three axioms of [HTT, Def. 6.2.2.1]: the maximal sieve is covering, covering sieves are stable under pullback, and covering is local on a cover.*

Definition 3.2.39 (Čech nerve of a morphism)**.** *Let $\mathcal{D}$ be an ∞-category with finite limits, and let*

$$u \colon d \to c$$

be a morphism in $\mathcal{D}$. The augmented Čech nerve *of u, denoted*

$$\check{C}^{+}(u)_{\bullet} \colon \Delta_{+}^{\mathrm{op}} \to \mathcal{D},$$

is the augmented simplicial object characterized by

$$\check{C}^{+}(u)_{-1} = c, \qquad \check{C}^{+}(u)_{n} \simeq d^{\times_c (n+1)} \quad (n \geq 0),$$

with face maps induced by the various projections and degeneracy maps induced by the diagonals. Equivalently, $\check{C}^{+}(u)_{\bullet}$ is the augmented simplicial object associated to u in the sense of [HTT, Def. 6.1.2.2 and Prop. 6.1.2.11]. We write

$$\check{C}(u)_{\bullet} := \check{C}^{+}(u)_{\bullet}|_{\Delta^{\mathrm{op}}}$$

for the underlying simplicial object.

Proposition 3.2.40 (Čech nerve of a covering family)**.** *Assume that C admits finite limits. Let*

$$\{f_i \colon c_i \to c\}_{i \in I}$$

be a family of morphisms in C, and let

$$u \colon \coprod_{i \in I} j(c_i) \to j(c)$$

be the induced morphism in $\mathcal{P}(C)$. Then the augmented Čech nerve $\check{C}^+(u)_\bullet$ in $\mathcal{P}(C)$ has terms

$$\check{C}(u)_n \simeq \coprod_{(i_0,\ldots,i_n) \in I^{n+1}} j(c_{i_0} \times_c \cdots \times_c c_{i_n}) \qquad (n \geq 0).$$

In particular,

$$\check{C}(u)_0 \simeq \coprod_i j(c_i),$$

$$\check{C}(u)_1 \simeq \coprod_{i,j} j(c_i \times_c c_j),$$

$$\check{C}(u)_2 \simeq \coprod_{i,j,k} j(c_i \times_c c_j \times_c c_k),$$

and so on.

Let

$$U \hookrightarrow j(c)$$

be the subobject corresponding to the sieve generated by the family $\{f_i\}_{i \in I}$. Then the induced map

$$\operatorname{colim} \check{C}(u)_\bullet \longrightarrow j(c)$$

identifies $\operatorname{colim} \check{C}(u)_\bullet$ *with U. Equivalently, $U \to j(c)$ is the (-1)-truncation of u; compare [HTT, Lem. 6.2.3.18] and the proof of [HTT, Prop. 6.2.4.6].*

Definition 3.2.41 (S-local objects)**.** *Let $\mathcal{C}$ be an ∞-category and let S be a collection of morphisms in $\mathcal{C}$. An object $Z \in \mathcal{C}$ is called* S-local *if, for every morphism*

$$s \colon X \to Y$$

in S, composition with s induces an equivalence of spaces

$$\mathbf{Map}_{\mathcal{C}}(Y, Z) \longrightarrow \mathbf{Map}_{\mathcal{C}}(X, Z)$$

[HTT, Def. 5.5.4.1].

Definition 3.2.42 (Sheaves on an ∞-site)**.** *Let C be a small ∞-category equipped with a Grothendieck topology J, and let*

$$j \colon C \to \mathcal{P}(C)$$

be the Yoneda embedding. By [HTT, Prop. 6.2.2.5], subobjects

$$U \hookrightarrow j(c)$$

are in bijection with sieves on $c \in C$*. Let* S *be the collection of those subobjects* $U \hookrightarrow j(c)$ *corresponding to covering sieves. A presheaf* $F \in \mathcal{P}(C)$ *is a* sheaf *if it is* S*-local; equivalently, for every covering sieve* $U \hookrightarrow j(c)$*, the restriction map*

$$\mathbf{Map}_{\mathcal{P}(C)}(j(c), F) \longrightarrow \mathbf{Map}_{\mathcal{P}(C)}(U, F)$$

is an equivalence [HTT, Def. 6.2.2.6]. Using Yoneda, this may be rewritten as

$$F(c) \simeq \mathbf{Map}_{\mathcal{P}(C)}(j(c), F) \xrightarrow{\sim} \mathbf{Map}_{\mathcal{P}(C)}(U, F),$$

so the value $F(c)$ *is identified with the space of homotopy-coherent matching data on the covering sieve* U*.*

Assume moreover that C *admits finite limits. Let*

$$\{f_i \colon c_i \to c\}_{i \in I}$$

be a family of morphisms generating a covering sieve on c, and let

$$u \colon \coprod_i j(c_i) \to j(c)$$

be the induced morphism in $\mathcal{P}(C)$*. If* $\check{C}(u)_\bullet$ *denotes its Čech nerve, then by the preceding proposition the associated covering sieve* $U \hookrightarrow j(c)$ *is equivalent to*

$$\operatorname{colim} \check{C}(u)_\bullet.$$

Consequently,

$$F(c) \simeq \mathbf{Map}_{\mathcal{P}(C)}(j(c), F) \xrightarrow{\sim} \mathbf{Map}_{\mathcal{P}(C)}(U, F) \simeq \lim_{[n]\in\Delta} \mathbf{Map}_{\mathcal{P}(C)}(\check{C}(u)_n, F),$$

so F *satisfies the usual Čech descent condition*

$$F(c) \simeq \lim\Big(\prod_i F(c_i) \rightrightarrows \prod_{i,j} F(c_i \times_c c_j) \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \prod_{i,j,k} F(c_i \times_c c_j \times_c c_k) \to \cdots\Big).$$

In other words, when C *has finite limits, a presheaf is a sheaf if and only if it sends every covering family to a limit diagram of the above form.*

The full subcategory of $\mathcal{P}(C)$ *spanned by sheaves with respect to* J *is denoted*

$$\operatorname{Shv}(C, J) \subseteq \mathcal{P}(C).$$

Definition 3.2.43 (∞-topos)**.** *An ∞-category $\mathcal{X}$ is an* ∞-topos *if there exists a small ∞-category C and an accessible left exact localization*

$$\mathcal{P}(C) \longrightarrow \mathcal{X}$$

[HTT, Def. 6.1.0.4]. Note that this data induces a Grothendieck topology J on C and a canonical equivalence $\operatorname{Shv}(C, J) \xrightarrow{\sim} \mathcal{X}$*. Equivalently, $\mathcal{X}$ is an ∞-topos if and only if it satisfies the ∞-categorical Giraud axioms of [HTT, Thm. 6.1.0.6]: it is presentable, colimits are universal, coproducts are disjoint, and every groupoid object is effective.*

Example 3.2.44. For every small ∞-category C, the presheaf ∞-category $\mathcal{P}(C)$ is an ∞-topos by [HTT, Def. 6.1.0.4]. If C is equipped with a Grothendieck topology J, then $\operatorname{Shv}(C, J)$ is an ∞-topos by [HTT, Prop. 6.2.2.7].

Definition 3.2.45. *In the above situation, the inclusion* $\operatorname{Shv}(C, J) \to \mathcal{P}(C)$ *is accessible and preserves limits, so by proposition 3.2.35 it admits a left adjoint* $\mathcal{P}(C) \to \operatorname{Shv}(C, J)$ *called the* sheafification*.*

Notation 3.2.46. If $S = (C, J)$ is a small ∞-site, we write

$$j_S \colon C \to \operatorname{Shv}(C, J)$$

for the composite of the Yoneda embedding with sheafification.

Definition 3.2.47. *For every small ∞-category C, a topology J on C is said to be* subcanonical *if every for every object $c \in C$, the Yoneda presheaf $j(c)$ is a sheaf for J. In this case, the functor $j_{(C,J)}$ as defined above coincides with the usual Yoneda embedding.*

Definition 3.2.48 (Geometric morphism)**.** *A* geometric morphism *from an ∞-topos $\mathcal{X}$ to an ∞-topos $\mathcal{Y}$ is a functor*

$$f_* \colon \mathcal{X} \to \mathcal{Y}$$

which admits a left exact left adjoint

$$f^* \colon \mathcal{Y} \to \mathcal{X}$$

[HTT, Def. 6.3.1.1].

Remark 3.2.49. The functor f^* is the *inverse image* functor and f_* is the *direct image* functor [HTT, Rem. 6.3.1.2]. Either determines the other up to contractible ambiguity, and the notation is arranged so that the superscripted asterisk denotes the left adjoint and the subscripted asterisk the right adjoint [HTT, Rem. 6.3.1.2].

Proposition 3.2.50. *Geometric morphisms are stable under composition: if*

$$\mathcal{X} \xrightarrow{f_*} \mathcal{Y} \xrightarrow{g_*} \mathcal{Z}$$

are geometric morphisms, then

$$g_* \circ f_* \colon \mathcal{X} \to \mathcal{Z}$$

is again a geometric morphism, with inverse image

$$(g_* \circ f_*)^* \simeq f^* \circ g^*$$

[HTT, Rem. 6.3.1.4].

Remark 3.2.51. To conclude this subsection, we will present two constructions of ∞-topoi associated to an initial ∞-topos, and describe their site presentation explicitly.

Definition 3.2.52 (Slice ∞-category)**.** *Let $\mathcal{X}$ be an ∞-category and let $U \in \mathcal{X}$. The* slice ∞-category *$\mathcal{X}_{/U}$ is the overcategory of U; its objects are morphisms $V \to U$ in $\mathcal{X}$ [HTT, Prop. 1.2.9.3 and Rem. 1.2.9.4].*

Proposition 3.2.53 (Slice ∞-topoi)**.** *If $\mathcal{X}$ is an ∞-topos and $U \in \mathcal{X}$, then $\mathcal{X}_{/U}$ is again an ∞-topos. The projection*

$$\pi_! \colon \mathcal{X}_{/U} \to \mathcal{X}$$

has a right adjoint

$$\pi^* \colon \mathcal{X} \to \mathcal{X}_{/U}, \qquad \pi^*(V) = V \times U \to U,$$

and π^* preserves colimits. Consequently, π^* has a further right adjoint, called the push-forward or Weil restriction

$$\pi_* \colon \mathcal{X}_{/U} \to \mathcal{X},$$

satisfying the universal property

$$\forall Z \in \mathcal{X}_{/U},\ \forall T \in \mathcal{X},\ \mathbf{Map}_{\mathcal{X}}(T, f_* Z) \simeq \mathbf{Map}_{\mathcal{X}_{/U}}(T \times U, Z)$$

so that

$$\pi_* \colon \mathcal{X}_{/U} \to \mathcal{X}$$

is a geometric morphism [HTT, Prop. 6.3.5.1].

Proposition 3.2.54 (Slice-site presentation)**.** *Let (C, J) be a small ∞-site with finite limits, $\mathcal{X} = \mathrm{Shv}(C, J)$, and let $X \in \mathcal{X}$. The* slice site *of X is the slice category $C_{/X}$, whose objects are couples (c, f) with $c \in C$ and $f : j(c) \to X$ in $\mathcal{X}$, equipped with the Grothendieck topology $J_{/X}$ defined as follows: a sieve R on an object*

$$u \colon j(c) \to X$$

of $\mathcal{X}_{/X}$ is covering if and only if its image under the forgetful functor

$$\nu_X \colon C_{/X} \to C$$

is a covering sieve on c for the topology J. Equivalently, R is covering if and only if it contains a family of morphisms

$$v_\alpha \colon j(c_\alpha) \to X$$

whose underlying family in C generates a J-covering sieve on c. This is the explicit slice form of the canonical topology of [HTT, Def. 6.2.4.1 and Prop. 6.2.4.2]. With this topology one obtains the standard slice presentation

$$\mathrm{Shv}(C, J)_{/X} \simeq \mathrm{Shv}(C_{/X}, J_{/X}).$$

Definition 3.2.55 (Arrow ∞-category)**.** *For any ∞-category $\mathcal{X}$, its* arrow ∞-category *is*

$$\mathrm{Arr}(\mathcal{X}) := \mathrm{Fun}(\Delta^1, \mathcal{X}).$$

Its objects are morphisms of $\mathcal{X}$, and its morphisms are homotopy-commutative squares.

Proposition 3.2.56. *If $\mathcal{X}$ is an ∞-topos, then the arrow ∞-category*

$$\mathrm{Arr}(\mathcal{X}) = \mathrm{Fun}(\Delta^1, \mathcal{X})$$

is again an ∞-topos. More generally, for every small ∞-category K, the cotensor $\mathrm{Mor}(K, \mathcal{X})$ *exists and is an ∞-topos [HTT, Prop. 6.3.4.9].*

Example 3.2.57 (Arrow-site presentation)**.** Let (C, J) be a small ∞-site with finite limits. The *arrow site* of (C, J) is the arrow category

$$\mathrm{Arr}(C) = \mathrm{Fun}(\Delta^1, C)$$

equipped with the *pointwise* Grothendieck topology J_{arr}, defined by the requirement that a sieve R on an arrow

$$u \colon c_0 \to c_1$$

is covering if and only if, for $i = 0, 1$, its image under the evaluation functor

$$ev_i \colon \mathrm{Arr}(C) \to C$$

is a covering sieve on c_i for the topology J. Equivalently, R is covering if and only if it contains a family of commutative squares

$$\begin{tikzcd}
d_{0,\alpha} \arrow[r] \arrow[d, "v_{0,\alpha}"'] & d_{1,\alpha} \arrow[d, "v_{1,\alpha}"] \\
c_0 \arrow[r, "u"'] & c_1
\end{tikzcd}$$

such that, for $i = 0, 1$, the family

$$\{v_{i,\alpha} \colon d_{i,\alpha} \to c_i\}_\alpha$$

generates a J-covering sieve on c_i. This is the explicit arrow form of the canonical topology associated to the pointwise functor

$$\mathrm{Arr}(C) \to \mathrm{Arr}(\mathrm{Shv}(C, J))$$

in the sense of [HTT, Def. 6.2.4.1 and Prop. 6.2.4.2]. With this topology one obtains the standard arrow-site presentation

$$\mathrm{Arr}(\mathrm{Shv}(C, J)) \simeq \mathrm{Shv}(\mathrm{Arr}(C), J_{\mathrm{arr}}).$$

3.2.3 Stable Infinity-Categories

While the ∞-topoi of the last section generalize the idea of ∞-categories of spaces, the notion of stable ∞-categories correspond to ∞-categories of quantities. More concretely, they correspond to the ∞-categorical version of Abelian categories as in [Gro57].

Definition 3.2.58. *A pointed ∞-category is an ∞-category admitting a zero object, i.e. an object* 0 *which is both initial and final. Let C be a pointed ∞-category. A* fiber *of a morphism $g \colon X \to Y$ is an object* $\mathrm{fib}(g)$ *fitting into a pullback square*

$$\begin{tikzcd}
\mathrm{fib}(g) \arrow[r] \arrow[d] & X \arrow[d, "g"] \\
0 \arrow[r] & Y,
\end{tikzcd}$$

and a cofiber *of g is an object* $\mathrm{cofib}(g)$ *fitting into a pushout square*

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\
0 \arrow[r] & \mathrm{cofib}(g).
\end{tikzcd}$$

Equivalently, one speaks of fiber sequences and cofiber sequences. More generally, we call triangles *the data of two maps $f : X \to Y$ and $g : Y \to Z$ and the choice a homotopy $g \circ f \simeq 0$, i.e. the data of the commutativity of the following square:*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\
0 \arrow[r] \arrow[ur, Leftrightarrow] & Z
\end{tikzcd}$$

Moreover, such a triangle is called exact *(resp.* co-exact*) if the above square is a pull-back (resp. push-out), i.e. if there is an identification* $X \simeq \mathrm{fib}(g)$ *(resp.* $Z \simeq \mathrm{cofib}(f)$*). [HA, Def. 1.1.1.6]*

Definition 3.2.59. *An ∞-category C is* stable *if it satisfies the following conditions:*

(i) C has a zero object;

(ii) every morphism in C admits a fiber and a cofiber;

(iii) a triangle in C is a fiber sequence if and only if it is a cofiber sequence.

This is Lurie's basic definition of stability [HA, Def. 1.1.1.9].

Proposition 3.2.60. *Let C be an ∞-category. The following are equivalent:*

(i) C is stable;

(ii) C is pointed, admits finite limits and finite colimits, and a square in C is a pullback if and only if it is a pushout.

In particular, in a stable ∞-category pullback squares and pushout squares coincide [HTT, Thm. 4.2.4.1 and Prop. 4.2.4.4].

Remark 3.2.61. The preceding characterization is often the most efficient one in practice. It shows that stability is an intrinsic property of the underlying ∞-category, rather than additional structure [HA, Def. 1.1.1.9 and Rem. 1.1.1.14].

Example 3.2.62. The ∞-category Sp of spectra is stable [HA, Ex. 1.1.1.11]. More generally, if A is an additive category, then the dg nerve $N_{\mathrm{dg}}(\mathrm{Ch}(A))$ of the category of chain complexes in A is stable [HA, Prop. 1.3.2.10]. In particular, derived ∞-categories of abelian categories furnish basic examples of stable ∞-categories [HA, Def. 1.3.2.7, Ex. 1.1.1.12, and Prop. 1.3.2.10].

Definition 3.2.63. *Let X be an object in a stable ∞-category C. We can define its loop and suspension objects as the following pull-backs/push-outs*

$$\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] & & X \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & X & & 0 \arrow[r] & \Sigma X \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

The objects ΩX and ΣX are usually denoted as $X[-1]$ and $X[1]$. The functors $\Sigma \dashv \Omega$ form an adjoint equivalence. More generally, the n-fold looping of X is denoted as $X[-n]$ and its n-fold suspension is denoted as $X[n]$. This defines functors $-[n]$ for $n \in \mathbb{Z}$, which satisfy $-[n] \circ -[m] = -[n+m]$ for all $n, m \in \mathbb{Z}$.

Proposition 3.2.64. *If C is a stable ∞-category, then its homotopy category hC carries a canonical triangulated structure; the translation functor is induced by suspension, and the distinguished triangles are those arising from cofiber sequences [HA, Thm. 1.1.2.14 and Rem. 1.1.2.15].*

Remark 3.2.65. The passage from C to hC recovers the classical triangulated formalism, but the stable ∞-category retains strictly more information: mapping spaces, functor ∞-categories, and higher coherences are still present before passing to the homotopy category [HA, Thm. 1.1.2.14]. This is one of the main advantages of working ∞-categorically rather than only triangulated-categorically.

Remark 3.2.66. Stable ∞-categories provide the natural setting for homological algebra. They supply intrinsic notions of fiber, cofiber, exact triangle, and derived functor, while remaining compatible with higher-categorical constructions such as limits, colimits, and functor categories [HA, Def. 1.1.1.9 and §1.3].

Definition 3.2.67. *Given an ∞-category $\mathcal{C}$ which admits finite limits (in particular, admitting a terminal object* 1*), we will denote by* $\mathrm{Sp}(\mathcal{C}) := \lim(\cdots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*)$ *the ∞-category of* spectrum objects in $\mathcal{C}$*, where $\mathcal{C}_* := 1/\mathcal{C}$ is the ∞-category of pointed objects in $\mathcal{C}$. [HA, Prop. 1.4.2.24 and Rmk. 1.4.2.25].*

Proposition 3.2.68. *For $\mathcal{C}$ an ∞-category with finite limits, the ∞-category* $\mathrm{Sp}(\mathcal{C})$ *is stable [HA, Cor. 1.4.2.17]. Moreover, the construction* Sp *canonically promotes to a functor* $\mathrm{Sp} : \mathrm{Pr}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}_{st}$ *where* $\mathrm{Pr}^{\mathrm{L}}_{st}$ *is the full sub-∞-category of presentable ∞-categories and left-adjoint functors spanned by stable presentable ∞-categories. It comes equipped with a canonical left-adjoint functor* $\Sigma^{\infty}_{+} : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$ *(whose right adjoint is denoted by* Ω_{∞}*), and is uniquely characterized by the following universal property: for any stable presentable ∞-category $\mathcal{A}$, composition with Σ^{∞}_{+} induces an equivalence*

$$\mathrm{Fun}^{L}(\mathrm{Sp}(\mathcal{C}), \mathcal{A}) \simeq \mathrm{Fun}^{L}(\mathcal{C}, \mathcal{A})$$

where Fun^{L} *denotes the space of left-adjoint functors [HA, Cor. 1.4.4.5].*

Remark 3.2.69. By the above proposition, the construction Sp provides the free stable presentable ∞-category of a presentable ∞-category. However, note that in general the canonical functor Σ^{∞}_{+} is neither fully faithful nor essentially surjective, as can be seen with $\Sigma^{\infty}_{+} : \mathcal{S} \to \mathrm{Sp}$ the canonical functor from spaces to spectra.

3.3 The Derived Affine Site

In this section, we will define the derived affine site, and consequently define derived stacks as sheaves on this site as in the previous section on ∞-topoi.

3.3.1 Homotopical Rings

Definition 3.3.1 (Homotopical rings). *A* homotopical ring *will mean one of the following objects.*

- *A simplicial commutative ring* $A \in \mathrm{sCAlg}$.
- *A connective* E_∞*-ring* $A \in \mathrm{CAlg}^{\mathrm{cn}}$.
- *If the base ring* k *satisfies* $k \supset \mathbb{Q}$*, a connective commutative dg-algebra* $A \in \mathrm{cdga}_k^{\leq 0}$.

The corresponding derived affine is denoted $\mathrm{Spec}(A)$*, and the derived affine site is the opposite* $\mathrm{dAff}^{\mathrm{der}} = (\textit{homotopical rings})^{\mathrm{op}}$*. See [LurieDAG5, Def. 4.1.1], [HAGII, Def. 1.1.0.11 and Def. 1.3.2.13], [HA, Prop. 7.1.4.11 and Prop. 7.1.4.20].*

Remark 3.3.2 (Discrete rings). Note that in each case, classical rings embed fully faithfully, and their essential images are called *discrete rings.* Equivalently, in each case there are homotopy groups functors $\pi_i : \mathrm{dRings} \to \mathrm{Set}$ for all $i \geq 0$, and discrete rings A are precisely those such that $\pi_i(A) \simeq 0$ for all $i \geq 1$.

For simplicial rings, see [LuriePhD, §3.1]. For connective E_∞-rings, see [HA, Rmk. 7.1.0.3] and [HA, Def. 7.1.3.17, Prop. 7.1.3.18]. For k-cdga's with $k \supset \mathbb{Q}$, see [HA, Def. 7.1.4.8, Rmk. 7.1.4.9, Props. 7.1.4.10 and 7.1.4.11].

Proposition 3.3.3 (Characteristic-zero rectification). *If* k *is a commutative* $\mathbb{Q}$*-algebra, then the* ∞*-categories of simplicial commutative* k*-algebras, connective* E_∞*-*k*-algebras, and connective commutative dg-*k*-algebras are canonically equivalent. Consequently, in characteristic zero one may pass freely between the simplicial and dg models when discussing connective derived affines. The analogous statement fails in general in positive characteristic. See [HA, Prop. 7.1.4.11, Prop. 7.1.4.20, and Warning 7.1.4.21].*

Remark 3.3.4. Also note that in the above case, homotopy groups π_i as simplicial rings or modules are equivalent to cohomology groups H^i as k-cdga's or k-dg-modules, and as such they can be interchanged in the following definitions and statements.

Definition 3.3.5 (A-modules). *Let* A *be a homotopical ring. The stable* ∞*-category of* A*-modules is denoted* $A\text{-}\mathrm{Mod}$ *or* Mod_A.

- *If* A *is simplicial commutative, simplicial* A*-modules form an* ∞*-category denoted by* Mod_A^{cn} *which is presentable but not stable in general. However, we define spectral* A*-modules by setting* $\mathrm{Mod}_A := \mathrm{Sp}(\mathrm{Mod}_A^{cn})$*, and the canonical functor* $\Sigma_+^\infty : \mathrm{Mod}_A^{cn} \to \mathrm{Mod}_A$ *is fully faithful in this case.*
- *If* A *is a connective cdga,* Mod_A *is obtained from (non-bounded) dg-modules over* A.
- *If* A *is a connective* E_∞*-ring,* Mod_A *is the* ∞*-category of* A*-module spectra.*

In every case Mod_A *is stable, presentable, and symmetric monoidal, with tensor product* $-\otimes_A-$ *and base-change functors* $-\otimes_A B\colon \mathrm{Mod}_A \to \mathrm{Mod}_B$ *attached to morphisms* $A \to B$*. See [HA, Def. 7.1.1.2, Prop. 7.1.1.4, Cor. 7.1.1.5], [HA, Prop. 4.5.1.4 and Thm. 4.5.2.1], and [HA, Cor. 4.2.3.7].*

Example 3.3.6 (Cotangent complexes)**.** For a morphism of homotopical rings $A \to B$, the *relative cotangent complex* $\mathbb{L}_{B/A} \in \mathrm{Mod}_B$ is the object corepresenting A-derivations from B into B-modules. The absolute cotangent complex is defined relative to the initial ring in the chosen context. The cotangent complex satisfies the transitivity fiber sequence

$$\mathbb{L}_A \otimes_A B \longrightarrow \mathbb{L}_B \longrightarrow \mathbb{L}_{B/A}$$

and base-change along homotopy pushouts. For a discrete commutative ring R, one recovers Kähler differentials by $\pi_0(\mathbb{L}_R) \simeq \Omega^1_R$. The cotangent complex is the basic example of an A-module encoding infinitesimal information. See [HAGII, Prop. 1.2.1.2, Def. 1.2.1.5, and Prop. 1.2.1.6], [HA, Def. 7.3.2.14, Def. 7.3.3.1, Rem. 7.3.3.2, Cor. 7.3.3.6, and Prop. 7.3.3.7], [HA, Lem. 7.4.3.8 and Prop. 7.4.3.9], and [LurieDAG14, Prop. 1.3.18, Cor. 1.3.19, and Rem. 1.3.21].

Proposition 3.3.7 (Postnikov-theoretic description)**.** *Connective homotopical rings admit Postnikov towers. Many geometric properties are controlled by the ordinary ring $\pi_0(A)$ together with the higher homotopy modules $\pi_i(A)$ and the cotangent complex. In particular, infinitesimal constructions are compatible with truncation, and nilpotent thickenings are expressed by square-zero extensions. See [HAGII, Lem. 2.2.1.1, Def. 1.2.1.7, Def. 2.2.4.3, and Prop. 2.2.4.4], [HA, Prop. 7.1.3.19, Cor. 7.4.1.27, and Cor. 7.4.1.28].*

3.3.2 Properties of Maps

Definition 3.3.8 (Flatness)**.** *A morphism $A \to B$ of connective homotopical rings is* flat *if $\pi_0(B)$ is flat over $\pi_0(A)$ and the canonical maps*

$$\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \longrightarrow \pi_i(B)$$

are isomorphisms for all $i \geq 0$. In the dg model over a characteristic-zero base this is equivalent to the corresponding derived flatness condition after passage to the simplicial model. See [HAGII, Def. 2.2.2.3 and Lem. 2.2.2.2], [HA, Def. 7.2.2.10 and Prop. 7.2.2.16], and [HA, Prop. 7.1.4.11 and Prop. 7.1.4.20].

Definition 3.3.9 (Finite presentation and smoothness conditions)**.** *Let $A \to B$ be a morphism of homotopical rings.*

- *It is* almost of finite presentation *if $\pi_0(A) \to \pi_0(B)$ is classically of finite presentation and the relative cotangent complex $\mathbb{L}_{B/A}$ is almost perfect as a B-module, i.e. its homotopy groups are finitely generated $\pi_0(B)$-modules and vanish for $i \gg 0$.*
- *It is* étale *if it is flat and $\pi_0(A) \to \pi_0(B)$ is classically étale. Equivalently, it is almost of finite presentation and $\mathbb{L}_{B/A} \simeq 0$.*
- *It is* smooth *if it is flat and $\pi_0(A) \to \pi_0(B)$ is classically smooth. Equivalently, it is almost of finite presentation and formally smooth.*

- *It is* quasi-smooth *if $\pi_0(A) \to \pi_0(B)$ is classically of finite presentation and $\mathbb{L}_{B/A}$ is almost perfect of Tor-amplitude ≤ 1; equivalently, in cohomological indexing, $\mathbb{L}_{B/A}$ has Tor-amplitude contained in* $[-1, 0]$.

These are the standard derived replacements of the corresponding classical classes of morphisms. See [HAGII, Prop. 2.2.2.4, Thm. 2.2.2.6, and Cor. 2.2.2.11], [HA, Def. 7.2.4.26, Prop. 7.2.4.27, and Thm. 7.4.3.18], [LuriePhD, Def. 3.4.7, Prop. 3.4.9, Cor. 3.4.10, and Def. 3.4.15].

Proposition 3.3.10 (Homotopy-group criterion for *étale* maps)**.** *For a morphism $A \to B$ of connective homotopical rings, the following criterion characterizes* étale *maps in the simplicial and spectral settings: the induced morphism $\pi_0(A) \to \pi_0(B)$ is classically* étale, *and the canonical maps*

$$\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \longrightarrow \pi_i(B)$$

are isomorphisms for all $i \geq 0$. This is one of the main reasons that étale *morphisms behave formally in derived algebraic geometry much as they do classically. See [HAGII, Thm. 2.2.2.6 and Cor. 2.2.2.11], [HA, Def. 7.5.0.4 and Thm. 7.5.0.6], and [LurieDAG5, Def. 4.3.3 and Prop. 4.3.9].*

Proposition 3.3.11 (Stability properties)**.** *The classes of flat,* étale*, smooth, and quasi-smooth morphisms are stable under equivalences, composition, and homotopy pullback. See [HAGII, Prop. 1.2.7.3, Prop. 1.2.8.2, and Prop. 1.3.6.3], [LuriePhD, Prop. 3.5.2, Prop. 3.5.5, and Prop. 3.5.8], and [HA, Remark 7.5.0.5].*

3.3.3 Homotopical Contexts

Definition 3.3.12 (Homotopical context)**.** *A* homotopical context *for derived algebraic geometry consists, in practice, of the following data:*

- *a category of derived affines* $\mathrm{dAff}^{\mathrm{der}}$*;*
- *a Grothendieck topology τ on* $\mathrm{dAff}^{\mathrm{der}}$*;*
- *a class* $\mathbf{P}$ *of admissible morphisms, typically smooth or* étale*, stable under equivalences, composition, and base change, and compatible with τ.*

In the model-categorical formulation of Toën-Vezzosi, this is encoded by a HAG-context $(C, C_0, \mathcal{A}, \tau, \mathbf{P})$; see [HAGII, Def. 1.1.0.11 and Def. 1.3.2.13].

Proposition 3.3.13 (Derived affine sites in the standard models)**.** *The standard choices are the following.*

- ***Derived algebraic geometry over a ring** k**:*** $\mathrm{dAff}^{\mathrm{der}} = (\mathrm{sCAlg}_k)^{\mathrm{op}}$ *with the* étale *topology, and smooth maps as the admissible class. This is the setting of geometric D^--stacks in [HAGII, Def. 2.2.2.12, Lem. 2.2.2.13, Def. 2.2.2.14, and Lem. 2.2.3.1].*

- ***Derived algebraic geometry in characteristic*** **0*:*** $\mathrm{dAff}^{\mathrm{der}} = (\mathrm{cdga}_k^{\leq 0})^{\mathrm{op}}$ *with the topology and admissible class transported from the above simplicial model. See [HAGII, Def. 2.3.1.3, Prop. 2.3.1.4, Lem. 2.3.4.1, and Def. 2.3.4.2].*
- ***Spectral algebraic geometry:*** $\mathrm{dAff}^{\mathrm{der}} = (\mathrm{CAlg}^{\mathrm{cn}})^{\mathrm{op}}$*, usually endowed with the* étale *topology and smooth maps as admissible class. See [HAGII, Def. 2.4.1.7, Def. 2.4.1.9, and Cor. 2.4.1.11]; see also [HA, Def. 7.5.0.4 and Thm. 7.5.0.6].*

Definition 3.3.14 (Prestack and derived stack)**.** *A* prestack *on the chosen affine site is a functor*

$$X\colon (\mathrm{dAff}^{\mathrm{der}})^{\mathrm{op}} \longrightarrow \mathcal{S}$$

with values in spaces. It is a derived stack *for the topology τ if it satisfies descent for τ-hypercovers; equivalently, for every τ-cover $U_\bullet \to U$, the canonical map*

$$X(U) \longrightarrow \lim X(U_\bullet)$$

is an equivalence. The representable prestacks $\mathrm{Spec}(A)$ *are stacks because the topologies under consideration are subcanonical in the sense of definition 3.2.47. See [HAGI, Def. 2.3.2], [HAGI, Cor. 3.4.7, Def. 3.4.8, and Def. 3.4.9], and [HAGII, Cor. 1.3.2.5].*

3.4 Derived Stacks and Quasi-Coherent Sheaves

In this thesis, we are working in derived algebraic geometry in characteristic 0, and using the étale topology. As a consequence, our model of derived rings is $\mathrm{dRings} := \mathrm{cdga}_k^{\leq 0}$ the ∞-category of connective k-cdgas (note that equivalences in this ∞-category correspond to quasi-isomorphisms, i.e. weak equivalences in the usual model structure on this category), where k is a fixed $\mathbb{Q}$-algebra, derived affines are $\mathrm{dAff} := \mathrm{dRings}^{\mathrm{op}}$, and we will set $\mathrm{dSt} := \mathrm{Shv}(\mathrm{dAff}, \tau_{et})$, which is a subcanonical topology in the sense of definition 3.2.47. Moreover, results of this section extend to derived prestacks as well, so we set $\mathrm{dPSt} := \mathrm{Fun}(\mathrm{dAff}^{\mathrm{op}}, \mathcal{S}) \supset \mathrm{dSt}$. In this section, we will define the standard geometric constructions on a derived (pre)stack X: its structure sheaf $\mathcal{O}_X$, its stable ∞-category of $\mathcal{O}_X$-modules, and finally quasi-coherent sheaves $\mathrm{QCoh}(X)$. In the next section, we will look at derived stacks satisfying geometric properties, i.e. derived Artin and Deligne-Mumford stacks, as they are better behaved than arbitrary derived (pre)stacks.

3.4.1 Structure Sheaf Ring

Definition 3.4.1 (Sheaves on a derived stack)**.** *Let X be a derived (pre)stack. We can consider its slice ∞-topos* $\mathrm{dSt}\,/X$ *of proposition 3.2.53, equivalent to* $\mathrm{Shv}(\mathrm{dAff}\,/X)$*, where* $\mathrm{dAff}\,/X$ *has objects affine morphisms $U = \mathrm{Spec}(A) \to X$ and coverings induced by coverings in* dAff*. A* sheaf on X *is simply an object of* $\mathrm{dSt}\,/X$*; more generally, for any presentable ∞-category $\mathcal{C}$, one may consider $\mathcal{C}$-valued sheaves on X as objects of*

$\mathrm{Shv}(\mathrm{dAff}/X;\mathcal{C})$*; see [HTT, Rmk. 6.3.5.10, Not. 6.3.5.16, Rmk. 6.3.5.17]. Note that, even when* X *does not satisfy the sheaf condition and is only a derived prestack, one can still study stacks over* X*.*

Remark 3.4.2. A *sheaf of rings* on X is thus an object of $\mathrm{Shv}(\mathrm{dAff}/X;\mathrm{dRings})$, the full sub-$\infty$-category of $\mathrm{Fun}(\mathrm{dAff}/X,\mathrm{dRings})$ spanned by functors satisfying the sheaf condition.

Definition 3.4.3 (The structure sheaf). *The* structure sheaf *of a derived (pre)stack* X *is the sheaf of commutative derived rings*

$$\mathcal{O}_X \in \mathrm{Fun}(\mathrm{dAff}/X,\mathrm{dRings})$$

defined on affine objects by

$$\mathcal{O}_X(\mathrm{Spec}(A) \xrightarrow{u} X) = A$$

This functor satisfies the sheaf condition, and thus $\mathcal{O}_X \in \mathrm{Shv}(\mathrm{dAff}/X;\mathrm{dRings})$*.*

3.4.2 Sheaves of Modules

Remark 3.4.4. Consider the functor $\mathrm{Mod} : \mathrm{dRings} \to \mathrm{Pr}^{\mathrm{L}}$ which maps a derived ring A to the presentable ∞-category $A\text{-}\mathrm{Mod}$ of A-modules, and associates to each map $f : A \to B$ the extension-of-scalars functor

$$f^* : A\text{-}\mathrm{Mod} \to B\text{-}\mathrm{Mod},$$

which lies in Pr^{L} since it is left adjoint to restriction of scalars.

We can form its total ∞-category $\mathcal{M}od$, whose objects are pairs (A, M) with A a derived ring and M an A-module, and a morphism $(A, M) \to (B, M')$ is the data of a map $A \to B$ together with a map $f^*M \to M'$ in $B\text{-}\mathrm{Mod}$. Equivalently, this is the coCartesian unstraightening of the functor Mod, equipped with its canonical projection $\mathcal{M}od \to \mathrm{dRings}$; see [HTT, Def. 5.5.3.1, Def. 5.5.3.2, Prop. 5.5.3.3(2)].

Definition 3.4.5 ($\mathcal{O}_X$-modules). *Let* X *be a derived (pre)stack. Let* Mod_X *be the* ∞-*category defined as the pull-back*

$$\begin{tikzcd}
\mathcal{M}od_X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}od \arrow[d] \\
\mathrm{dAff}/X \arrow[r, "\mathcal{O}_X"'] & \mathrm{dRings}
\end{tikzcd}$$

More concretely, objects of $\mathcal{M}od_X$ *are pairs* (u, M) *with* $u : \mathrm{Spec}(A) \to X$ *and* $M \in A\text{-}\mathrm{Mod}$*, and* $p : \mathcal{M}od_X \to \mathrm{dAff}/X$ *maps such an object to* u*. The* ∞*-category of* $\mathcal{O}_X$-modules *is the* ∞*-category of sections of* p*, i.e. the full subcategory of* $\mathrm{Fun}(\mathrm{dAff}/X, \mathrm{Mod}_X)$ *spanned by functors* $s : \mathrm{dAff}/X \to \mathrm{Mod}_X$ *such that* $p \circ s \simeq \mathrm{id}_{\mathrm{dAff}/X}$*.*

Equivalently, its objects can be described as the data:

- *For each* $u : \mathrm{Spec}(A) \to X$, *of an* A*-module* M_u*;*
- *For each* $u : \mathrm{Spec}(A) \to X$, $v : \mathrm{Spec}(B) \to X$ *and* $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *over* X, *of a transition map* $M_f : f^*M_u \to M_v$.

Equivalently, this identifies $\mathcal{O}_X\text{-}\mathrm{Mod}$ *with the lax limit*

$$\mathcal{O}_X\text{-}\mathrm{Mod} \simeq \lim^{\mathrm{lax}}_{(\mathrm{Spec}\, A \to X) \in (\mathrm{dAff}\,/X)^{\mathrm{op}}} \mathrm{Mod}_A\,.$$

As a module category over a sheaf of rings on an ∞*-topos,* $\mathcal{O}_X\text{-}\mathrm{Mod}$ *is a presentable stable* ∞*-category, naturally symmetric monoidal for the relative tensor product* $- \otimes_{\mathcal{O}_X} -$*; see [LurieDAG8, Prop. 2.1.3]. The straightening/unstraightening input behind the construction is [HTT, Def. 5.5.3.1, Def. 5.5.3.2, Prop. 5.5.3.3(2), Prop. 5.5.3.13, Cor. 5.5.3.14].*

Definition 3.4.6 (Pullback and pushforward)**.** *A morphism of derived (pre)stacks* $f\colon X \to Y$ *induces a morphism of sites* $\mathrm{dAff}\,/X \to \mathrm{dAff}\,/Y$ *by composition along* f*. As such, for any presentable* ∞*-category* $\mathcal{C}$*, this induces a functor*

$$f^{-1} : \mathrm{Shv}(\mathrm{dAff}\,/Y; \mathcal{C}) \to \mathrm{Shv}(\mathrm{dAff}\,/X; \mathcal{C}).$$

More concretely, given $F \in \mathrm{Shv}(\mathrm{dAff}\,/Y; \mathcal{C})$*, the sheaf* $f^{-1}F \in \mathrm{Shv}(\mathrm{dAff}\,/X; \mathcal{C})$ *is given by*

$$(\mathrm{Spec}(A) \to X) \longmapsto F(\mathrm{Spec}(A) \to X \to Y).$$

Moreover, there is a canonical map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$*. At the level of* $\mathcal{O}$*-modules, one defines the pullback functor* $f^* : \mathcal{O}_Y - \mathrm{Mod} \to \mathcal{O}_X - \mathrm{Mod}$ *by*

$$f^*(M) := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}M.$$

Since module categories over sheaves of rings are presentable and f^* *preserves small colimits, it admits a right adjoint*

$$f_*\colon\ \mathrm{Mod}_{\mathcal{O}_X}(X) \longrightarrow \mathrm{Mod}_{\mathcal{O}_Y}(Y);$$

see [LurieDAG8, Prop. 2.1.3] and [HTT, Cor. 5.5.2.9].

3.4.3 Quasi-Coherent Sheaves

Definition 3.4.7 (Definition by straightening)**.** *Coming back to the diagram of definition 3.4.5,*

$$\begin{tikzcd}
\mathcal{M}od_X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{M}od \arrow[d] \\
\mathrm{dAff}\,/X \arrow[r, "\mathcal{O}_X"'] & \mathrm{dRings}
\end{tikzcd}$$

we define QCoh(X) *to be the full subcategory of* $\mathcal{O}_X$-Mod *spanned by the* coCartesian *sections of* p*. Equivalently, an object of* QCoh(X) *is an* $\mathcal{O}_X$*-module* M *whose transition maps are equivalences.*

Equivalently, this presentation identifies QCoh(X) *with the limit*

$$\mathrm{QCoh}(X) \simeq \lim_{(\operatorname{Spec} A \to X) \in (\mathrm{dAff}\,/X)^{\mathrm{op}}} \mathrm{Mod}_A\,.$$

In the functorial language of DAG VIII, this is the category of coCartesian sections associated to the diagram $A \mapsto A$-Mod*; see [LurieDAG8, Def. 2.7.8, Not. 2.7.11, Rmk. 2.7.12, Prop. 2.7.18]. As a consequence,* QCoh(X) *is stable and admits small colimits, and presentable; see [LurieDAG8, Prop. 2.7.17 and Prop. 2.7.18]. It also carries a canonical symmetric monoidal structure induced from tensor product on affine module categories; see [LurieDAG8, Prop. 2.3.14 and Prop. 2.7.18]. The same definition and formal consequences hold for any derived prestack, see [GR17, Ch. 2, Sect. 1.1.4; Ch. 3, Sects. 1.1.3 and 1.1.4; Ch. 1, Sects. 5.1.8, 10.3.3 and 10.3.4].*

Remark 3.4.8. Note that for A a derived ring, while $\mathcal{O}_{\mathrm{Spec}(A)}$-Mod is not, in general, equivalent to A-Mod, one has

$$\mathrm{QCoh}(\mathrm{Spec}(A)) \simeq A\text{-}\mathrm{Mod};$$

see [LurieDAG8, Prop. 2.3.11, Rmk. 2.7.10 and Prop. 2.7.18]. This is a motivation for considering QCoh instead of $\mathcal{O}$ − Mod on derived (pre)stacks X to capture complexes that behave as modules on X more similarly to the affine case of A − Mod. This affine normalization is unchanged when affines are regarded as derived prestacks; see [GR17, Ch. 3, Sects. 1.1.2 and 1.1.4].

Proposition 3.4.9. *For* X *a derived (pre)stack, there is a canonical fully faithful inclusion* QCoh$(X) \hookrightarrow \mathcal{O}_X$-Mod*. Since* QCoh$(X)$ *is presentable and, in the big-site model above, the inclusion preserves small colimits, it admits a right adjoint*

$$(-)^{\mathrm{QCoh}} : \mathcal{O}_X\text{-}\mathrm{Mod} \to \mathrm{QCoh}(X)$$

called the quasi-coherator*; compare [LurieDAG8, Prop. 2.7.17 and Prop. 2.7.18] and [HTT, Cor. 5.5.2.9].*

Proposition 3.4.10 (Formal properties of QCoh(X))**.** *For every derived (pre)stack* X*:*

- QCoh(X) *is stable and presentable;*
- QCoh(X) *carries a symmetric monoidal structure induced from tensor product on affine module categories, or equivalently as a restriction of the tensor product on* $\mathcal{O}_X$ − Mod*, which preserves quasi-coherent sheaves;*

- *A morphism $f\colon X \to Y$ induces a colimit-preserving pullback functor*

$$f^*\colon \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X);$$

  equivalently presented as the restriction of the pullback of $\mathcal{O}$-modules, which preserves quasi-coherent sheaves;
- *Since f^* preserves colimits between presentable ∞-categories, it admits a right adjoint f_*, but note that it is not the restriction of the pushforward of $\mathcal{O}$-modules, which does not preserve quasi-coherent sheaves in general, but its composition with the quasi-coherator $(-)^{QCoh}$;*
- *If $X = \mathrm{Spec}(A)$ is affine, then $\mathrm{QCoh}(X) \simeq \mathrm{Mod}_A$.*

See [LurieDAG8, Prop. 2.3.14, Prop. 2.5.1, Not. 2.7.11, Prop. 2.7.17, Prop. 2.7.18, Rmk. 2.7.10] and [HTT, Cor. 5.5.2.9]. The same formal properties hold for arbitrary derived prestacks, with f_* understood as the possibly discontinuous right adjoint of f^*; see [GR17, Ch. 2, Sect. 1.1.4; Ch. 3, Sects. 1.1.3 and 1.1.4, 2.1.1, 3.1.3, 3.2.1 and 3.2.2; Ch. 1, Sects. 5.1.8, 10.3.3 and 10.3.4].

Proposition 3.4.11 (Descent for QCoh)**.** QCoh *satisfies étale descent, in the sense that if $U_\bullet \to X$ is an étale hypercover (for example the Čech nerve of an étale cover $U \to X$), then*

$$\mathrm{QCoh}(X) \simeq \lim \mathrm{QCoh}(U_\bullet).$$

See [LurieDAG8, Prop. 2.7.14]. This descent statement also holds in the prestack setting, and stackification does not change QCoh*; see [GR17, Ch. 3, Thm. 1.3.4 and Cors. 1.3.7 and 1.3.11]. As a consequence, since* QCoh *is defined by right Kan extension and it satisfies étale descent, then more generally by [HTT, Prop. 5.5.4.20] we deduce that it sends arbitrary colimits to limits, and not only étale covers.*

Definition 3.4.12. *For X a derived (pre)stack, the full subcategory of $\mathrm{QCoh}(X)$ spanned by the objects which are dualizable with respect to the tensor product $\otimes$, in the sense of [HA, Def. 4.6.1.7, Rmk. 4.6.1.12] will be called* perfect complexes *on X. The full subcategory itself is denoted by $\mathrm{Perf}(X) \subset \mathrm{QCoh}(X)$. For derived prestack Y, see the definition $\mathrm{Perf}(Y) := \mathrm{QCoh}(Y)_{\mathrm{perf}}$ in [GR17, Ch. 3, Sect. 3.6.1].*

Proposition 3.4.13. *For X a derived (pre)stack, $\mathrm{Perf}(X)$ satisfies the following formal properties:*

- $\mathrm{Perf}(X)$ *is a stable category. In particular, it admits finite limits and finite colimits, but it is not closed under arbitrary limits and colimits, so it is not presentable. [HAGII, Def. 1.2.3.6 and Prop. 1.2.3.7(5), (6)][LurieDAG8, Props. 2.7.26 and 2.7.28]*
- $\mathrm{Perf}(X)$ *carries a symmetric monoidal structure induced by restriction of $\otimes$ on $\mathrm{QCoh}(X)$ which preserves perfect complexes. [HAGII, Prop. 1.2.3.7(1)][LurieDAG8, Notation 2.7.27 and Prop. 2.7.28][HA, Def. 4.6.1.7 and Rmk. 4.6.1.12]*

- *A morphism of derived stacks* $f : X \to Y$ *induces a pull-back functor* $f^* : \mathrm{Perf}(Y) \to \mathrm{Perf}(X)$*, as a restriction of the pull-back of quasi-coherent sheaves, which does preserve perfect objects. [LurieDAG8, Notation 2.7.11, Prop. 2.7.20(7), Rmk. 2.7.22, and Prop. 2.7.18]*
- *However, in general,* f^* *does not admit a right adjoint* f_**, because the push-forward functor need not preserve perfect complexes (*f_* *preserves perfect complexes only when* f *satisfies further assumptions, which are usually related to properness of* f*). [LuriePhD, Rmk. 5.5.3 and Prop. 5.5.5(6)]*
- *If* $X = \mathrm{Spec}(A)$ *is affine, then* $\mathrm{Perf}(X)$ *is the* ∞*-category of perfect* A*-modules in the usual sense. In particular, in this case, an object of* $A - \mathrm{Mod}$ *is perfect if and only if it is compact, and also if and only if it is of bounded Tor-amplitude. [HAGII, Def. 1.2.3.6][LurieDAG8, Rmk. 2.7.10 and Prop. 2.7.28]*
- *The functor* Perf *satisfies descent, i.e. if* $U_\bullet \to X$ *is a hypercover (for example the Čech nerve of a cover* $U \to X$*), then*

  $$\mathrm{Perf}(X) \simeq \lim \mathrm{Perf}(U_\bullet)$$

  [HAGII, Cor. 1.3.7.4 and Def. 1.3.7.5][LurieDAG8, Props. 2.7.14, 2.7.18, 2.7.20(7), 2.7.26, 2.7.28, and Rmk. 2.7.16]

For the prestack analogues of the definition, pullback functoriality, affine test, and descent assertions, see the definition and properties of $Y \mapsto \mathrm{QCoh}(Y)_{\mathrm{perf}}$ *in [GR17, Ch. 3, Sects. 3.6.1 and 3.6.2, Cors. 3.6.4 and 3.6.5, and Lem. 3.6.7].*

Remark 3.4.14. For X a derived stack or derived prestack, the formal properties listed above do not by themselves allow the existence of relevant geometric quasi-coherent sheaves on X, most notably of the cotangent complex $\mathbb{L}_X$. Existence of cotangent complexes and deformation theory need further hypothesis on X, typically such as being derived Deligne-Mumford or derived Artin, which motivates their introduction in the next section.

3.5 Geometric Stacks

In this section, we will introduce derived Artin and derived Deligne-Mumford stacks. We will then describe what are cotangent complexes and obstruction theories, and their existence on derived Artin and derived Deligne-Mumford stacks. Finally, we will state a representability criterion that characterize derived Artin stacks and relate classical algebraic geometry with derived algebraic geometry.

3.5.1 Derived Artin stacks and derived Deligne-Mumford stacks

In this subsection, we will define derived Artin and derived Deligne-Mumford stacks and record their basic stability properties under truncation and base change.

Definition 3.5.1 (Derived Artin n-stacks)**.** *Let X be a derived stack. One defines by induction on n the property X* is derived Artin n-stack *in the following way.*

X is a derived Artin (-1)-stack if it is a derived affine.

For $n \geq 0$, X is a derived Artin n-stack if the following conditions hold:

- *the diagonal*
$$X \longrightarrow X \times X$$
is a relative derived Artin $(n-1)$-stack, meaning that for every derived affine $U \to X \times X$, the fiber product $X \times_{X \times X} U$ is a derived Artin $(n-1)$-stack;
- *there exists a covering*
$$\{U_i \to X\}_{i \in I}$$
with each U_i affine, each $U_i \to X$ smooth, and each $U_i \to X$ a relative derived Artin $(n-1)$-stack.

As above, a morphism $f \colon X \to Y$ is a relative derived Artin n-stack *if for every derived affine $U \to Y$, the pullback $X \times_Y U$ is a derived Artin n-stack.*

A derived stack X (resp. a morphism $f : X \to Y$) is said to be a derived Artin stack (resp. a relative derived Artin stack) if there exists an n such that X (resp. f) is a derived Artin n-stack (resp. relative derived Artin n-stack).

Note that compared to our reference, this is the general inductive notion of geometricity specialized to smooth morphisms; see [HAGII, Def. 1.3.3.1, Prop. 2.2.3.2].

Definition 3.5.2 (Derived Deligne-Mumford n-stacks)**.** *A derived stack X is a* derived Deligne-Mumford n-stack *if the same inductive definition is used, with* étale *morphisms in place of smooth morphisms.*

Thus, for $n = -1$, X is affine. For $n \geq 0$, one requires:

- *the diagonal $X \to X \times X$ to be a relative derived Deligne-Mumford $(n-1)$-stack;*
- *the existence of a covering $\{U_i \to X\}$ with each U_i affine, each $U_i \to X$ étale, and each $U_i \to X$ a relative derived Deligne-Mumford $(n-1)$-stack.*

A morphism $f \colon X \to Y$ is a relative derived Deligne-Mumford n-stack *if for every derived affine $U \to Y$, the pullback $X \times_Y U$ is a derived Deligne-Mumford n-stack.*

A derived stack X (resp. a morphism $f : X \to Y$) is said to be a derived Deligne-Mumford stack (resp. a relative derived Deligne-Mumford stack) if there exists an n such that X (resp. f) is a derived Deligne-Mumford n-stack (resp. relative derived Deligne-Mumford n-stack).

Note that since every étale morphism is smooth, every (relative) derived Deligne-Mumford n-stack is a (relative) derived Artin n-stack. See [HAGII, Def. 1.3.3.1, Cor. 2.2.2.11, Cor. 2.2.5.6].

Remark 3.5.3. Note that relative derived Artin or Deligne-Mumford stacks are stable under homotopy pullback and composition. Moreover, derived Artin or Deligne-Mumford n-stacks are stable under homotopy pullback. In particular, derived Artin and derived Deligne-Mumford stacks are stable under base change.

Additionaly, the truncation functor (given by restriction along the inclusion of discrete rings in derived rings)

$$t_0\colon \mathrm{dSt}_k \longrightarrow \mathrm{St}_k$$

preserves n-geometricity and flat, smooth, and étale morphisms. Hence the truncation of a derived Artin stack is a classical Artin stack, and the truncation of a derived Deligne-Mumford stack is a classical Deligne-Mumford stack. Conversely, the inclusion of classical stacks into derived stacks preserves geometricity and these classes of morphisms. See [HAGII, Prop. 1.3.3.3, Cor. 1.3.3.5, Prop. 2.2.4.4].

3.5.2 Cotangent complexes and infinitesimal theory

Let us now introduce relative derivations, cotangent complexes, and obstruction theories in derived algebraic geometry.

Definition 3.5.4 (Relative derivations)**.** *Let* $f\colon X \longrightarrow Y$ *be a morphism of derived stacks, let* A *be a connective* k*-cdga, and let* $x\colon \mathrm{Spec}(A) \longrightarrow X$ *be an* A*-point of* X*. For an* A*-dg-module* M*, write* $\mathrm{Spec}(A \oplus M) \longrightarrow \mathrm{Spec}(A)$ *for the trivial square-zero extension. The simplicial set of* relative derivations of X over Y at x with values in M *is*

$$\mathrm{Der}_{X/Y}(x, M) := \mathbf{Map}_{\mathrm{Spec}(A)/\,\mathrm{dSt}_k\,/Y}\Big(\mathrm{Spec}(A \oplus M), X\Big).$$

Equivalently, $\mathrm{Der}_{X/Y}(x, -)$ *is the homotopy fiber of*

$$\mathrm{Der}_X(x, -) \longrightarrow \mathrm{Der}_Y(f \circ x, -).$$

This is introduced in the paragraph preceding [HAGII, Def. 1.4.1.14].

Definition 3.5.5 (Relative cotangent complexes)**.** *Let*

$$f\colon X \to Y$$

be a morphism of derived stacks, let A *be a derived ring, and let*

$$x\colon \mathrm{Spec}(A) \to X$$

be an A*-point. One says that* f *has a* relative cotangent complex at x *if the functor*

$$\mathrm{Der}_{X/Y}(x, -)$$

is corepresentable by an A-module $\mathbb{L}_{X/Y,x}$. One says that f has a global relative cotangent complex *if these local complexes exist, assemble into an $\mathcal{O}_X$-module, and satisfy the usual base-change compatibility, and thus define a quasi-coherent sheaf. One says that a derived stack X admits a cotangent complex if the terminal map $X \to \bullet$ admits a cotangent complex. See [HAGII, Def. 1.4.1.14 and Def. 1.4.1.15].*

Proposition 3.5.6 (Affine case)**.** *Let $f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ be the morphism of affine derived stacks associated to a morphism of derived rings $A \to B$. Then f has a relative cotangent complex, and at the identity point*

$$\mathrm{id}_{\operatorname{Spec}(B)}\colon \operatorname{Spec}(B) \to \operatorname{Spec}(B)$$

it is canonically identified with the usual relative cotangent complex

$$\mathbb{L}_{\operatorname{Spec}(B)/\operatorname{Spec}(A),\,\mathrm{id}} \simeq \mathbb{L}_{B/A}$$

in $B - \mathrm{Mod}$; under the equivalence $\mathrm{QCoh}(\operatorname{Spec}(B)) \simeq B\text{-}\mathrm{Mod}$, the corresponding quasi-coherent sheaf is $\mathbb{L}_{B/A}$. Equivalently, for every B-module M,

$$\mathrm{Der}_{\operatorname{Spec}(B)/\operatorname{Spec}(A)}(\mathrm{id}, M) \simeq \mathbb{R}\operatorname{Hom}_B(\mathbb{L}_{B/A}, M).$$

This follows from [HAGII, Prop. 1.2.1.2, Rmk. 1.2.1.4, Def. 1.2.1.5, Lem. 1.4.1.16(4)].

Proposition 3.5.7 (Transitivity and base change)**.** *Cotangent complexes satisfy two basic formal properties.*

First, if both F and G admit cotangent complexes, then every morphism $f\colon F \to G$ admits a relative cotangent complex, and for every affine point $x\colon \operatorname{Spec}(A) \to F$ there is a natural homotopy cofiber sequence

$$\mathbb{L}_{G,x} \longrightarrow \mathbb{L}_{F,x} \longrightarrow \mathbb{L}_{F/G,x}.$$

Second, relative cotangent complexes are stable under base change: for any morphism $H \to G$, the pullback $F \times_G H \to H$ admits a relative cotangent complex, identified with the base change of $\mathbb{L}_{F/G}$.

Hence cotangent complexes may be computed locally on an atlas, and are compatible with the descent formalism. See [HAGII, Lem. 1.4.1.16].

Definition 3.5.8 (Morphisms with an obstruction theory)**.** *Let*

$$f\colon X \longrightarrow Y$$

be a morphism of derived stacks. One says that f is infinitesimally cartesian *if for every connective k-cdga A, every connective A-dg-module M, and every derivation*

$$d \in \pi_0 \operatorname{Der}(A, M)$$

with associated square-zero extension

$$A \oplus_d \Omega M \longrightarrow A,$$

the natural square

$$\begin{tikzcd}
X(A \oplus_d \Omega M) \arrow[r] \arrow[d] & Y(A \oplus_d \Omega M) \arrow[d] \\
X(A) \times_{X(A \oplus M)} X(A) \arrow[r] & Y(A) \times_{Y(A \oplus M)} Y(A)
\end{tikzcd}$$

is homotopy cartesian.

One says that f has an obstruction theory *if:*

- *f has a global relative cotangent complex;*
- *f is infinitesimally cartesian.*

This is the relative notion of [HAGII, Def. 1.4.2.2].

Proposition 3.5.9 (Obstructions and square-zero extensions)**.** *Let $f \colon F \to G$ be a morphism with obstruction theory. Fix an affine point $y \colon \operatorname{Spec}(A) \to F$, a connected A-module M, and a square-zero extension*

$$A \oplus_d \Omega M \longrightarrow A.$$

Given a compatible point

$$x \in F(A) \times_{G(A)} G(A \oplus_d \Omega M),$$

there is a canonical obstruction class

$$\alpha(x) \in \mathbf{Map}_{A\text{-}\mathrm{Mod}}(\mathbb{L}_{F/G,y}, M),$$

and the space of lifts of x to a point of $F(A \oplus_d \Omega M)$ is the space of paths from $\alpha(x)$ to 0*. See [HAGII, Prop. 1.4.2.6].*

Proposition 3.5.10 (Existence for derived Artin stacks)**.** *In the HAG II setting, the étale topology together with smooth morphisms satisfies Artin's conditions. Hence every (relative) derived Artin stack has an obstruction theory, and therefore a cotangent complex. In particular, the same is true for derived Deligne-Mumford stacks. See [HAGII, Def. 1.4.3.1, Thm. 1.4.3.2, Prop. 2.2.3.2, Cor. 2.2.3.3].*

Remark 3.5.11. The cotangent complex can be described more concretely for relative derived Artin stacks. Let $p : Z \to X$ be an arbitrary map of derived stacks. Given a commutative square

$$\begin{tikzcd}
\mathrm{Spec}(B) \arrow[r] \arrow[d] & Z \arrow[d] \\
\mathrm{Spec}(A) \arrow[r] & X
\end{tikzcd}$$

i.e. an object of $\mathrm{Arr}(\mathrm{dAff})/p$, we can associate $\mathbb{L}_{B/A} \in B-\mathrm{Mod}$, and there are canonical transition maps. This defines an object $\mathbb{L}^{\mathrm{big}}_{Z/X}$ in the lax-limit of the ∞-categories $B-\mathrm{Mod}$ over such squares, which can be showed to be equivalent to $\mathcal{O}_Z-\mathrm{Mod}$. This object does represent derivations on Z relative to X in a suitable sense, but fails to be quasi-coherent in general. By applying the quasi-coherator $(-)^{qcoh} : \mathcal{O}_Z-\mathrm{Mod} \to \mathrm{QCoh}(Z)$, we do obtain an object $\mathbb{L}^{\mathrm{big,qcoh}}_{Z/X} \in \mathrm{QCoh}(Z)$, but this object then no longer represents derivations on Z relative to X in general. This can be seen by the fact that some derived stacks do not admit cotangent complexes, such as the moduli stack $\mathcal{M}_{Dcoh}$ in [Kha25, Rmk. 2.23], so the above construction has no hope of representing derivations in general. Moreover, even when $Z \to X$ admits a geniune quasi-coherent cotangent complex $\mathbb{L}_{Z/X}$, in which case there is a canonical map $\mathbb{L}_{Z/X} \to \mathbb{L}^{\mathrm{big,qcoh}}_{Z/X}$, it needs not be an equivalence in general. However, if $Z \to X$ is a relative derived Artin stack locally of finite presentation, then as above $\mathbb{L}_{Z/X}$ exists, and moreover by [TV25a, Thm. A.5.0.1] the canonical map $\mathbb{L}_{Z/X} \to \mathbb{L}^{\mathrm{big,qcoh}}_{Z/X}$ is an equivalence. More generally, we expect this result to hold for an arbitrary map of derived prestacks $Z \to X$ such that $\mathbb{L}_{Z/X}$ exists in $\mathrm{QCoh}(Z)$ and is bounded above (i.e. eventually connective, or equivalently it means that $\mathbb{L}_{Z/X} \in \mathrm{QCoh}^{-}(Z)$ in the sense of [Mon21, Thm. 0.2]). See [CS26, Thm. 2.7] which shows that their global sections coincide in this case, but we will not investigate this question in this thesis.

Remark 3.5.12 (Classical stacks inside derived geometry)**.** If X is a classical Artin stack, then its image in derived stacks is a derived Artin stack, therefore it has a canonical obstruction theory and a cotangent complex.

Moreover, for any derived Artin or Deligne-Mumford stack F, the natural morphism

$$t_0(F) \longrightarrow F$$

is relatively affine and behaves like a closed immersion, in the sense that after pullback to an affine $\mathrm{Spec}(A) \to F$, it becomes the natural closed embedding

$$\mathrm{Spec}(\pi_0 A) \longrightarrow \mathrm{Spec}(A).$$

This identifies a derived stack as an infinitesimal thickening of its classical truncation. See [HAGII, Cor. 2.2.4.5, Prop. 2.2.4.7].

Theorem 3.5.13 (Representability criterion, after Lurie)**.** *Let F be a derived stack. Then the following are equivalent:*

1. *F is a derived Artin n-stack;*
2. *the following hold:*
   - *(a) the truncation $t_0(F)$ is a classical Artin $(n+1)$-stack;*
   - *(b) F has an obstruction theory relative to connected modules;*
   - *(c) for every derived ring A, the canonical map*

     $$F(A) \longrightarrow \lim_k F(A_{\leq k})$$

     is an equivalence.

The third condition is the convergence, or nilcompleteness, condition: F is determined by its restrictions to the Postnikov truncations of A. See [HAGII, Thm. C.0.9].

CHAPTER 4

DERIVED FOLIATIONS

4.1 INTRODUCTION

A classical foliation on a smooth algebraic variety X may be described either by a subbundle $T_{\mathcal{F}} \subset T_X$ stable by the Lie bracket or, dually, by a differential ideal in the de Rham algebra Ω^*_X. The second description is the starting point of derived foliations. Instead of encoding a foliation as a distribution inside the tangent bundle, one encodes it by a quotient of the de Rham algebra carrying the de Rham differential along the leaves. In derived algebraic geometry this quotient is naturally expressed as a *graded mixed* commutative dg-algebra, and a derived foliation on a derived scheme or stack X is morally the data of such a graded mixed cdga $\mathbf{DR}(\mathcal{F})$ together with a morphism $\mathbf{DR}(X) \longrightarrow \mathbf{DR}(\mathcal{F})$, coming from an *anchor map* $a : \mathbb{L}_X \to \mathbb{L}_{\mathcal{F}}$, satisfying finiteness and freeness conditions on the underlying graded object. The weight-one piece of $\mathbf{DR}(\mathcal{F})$ is the cotangent complex $L_{\mathcal{F}}$ of the foliation, so derived foliations are controlled by a perfect complex rather than by an ordinary vector bundle.

A quasi-smooth derived foliation on a smooth variety has a classical truncation: its cotangent complex determines a differential ideal in the classical cotangent Ω^*_X, hence a singular foliation on the classical truncation. In this way, derived foliations refine certain singular foliations by equipping them with extra higher data. However, unlike ordinary derived enhancements of schemes, not every classical singular foliation admits a derived enhancement, even locally. A derived foliation should thus be viewed as a singular foliation together with extra structure. However, many examples of singular foliations in classical algebraic geometry do admit such a derived enhancement.

Quasi-smooth derived foliations form the class closest to ordinary singular foliations. Their cotangent complexes are two-term perfect complexes, so they retain enough rigidity to support a geometric theory while still allowing genuine singularities. On the smooth locus they resemble ordinary foliations; on the singular locus they record the defect of smoothness in derived terms. Much of the initial theory, as presented in [TV23a; TV20], is devoted to understanding how this extra structure controls local behavior and how it improves the formal properties of singular foliations.

A central theme is integrability. In the classical smooth setting, Frobenius theory identifies involutive distributions with foliations by leaves. For singular algebraic foliations, integrability is more delicate. Derived foliations exhibit stronger formal integrability properties: by [BMN25a], one can associate formal leaves and formal leaf spaces, whose behavior on formal completions are controlled by dg-Lie theoretic data. In this sense, all derived foliations are formally integrable, so these objects retain the classical intuition of longitudinal and transverse geometry, but in a homotopy-coherent form adapted to singular and derived contexts. This is one of the main structural advantages of the theory.

For more information and examples of derived foliations, see the reference book [TV25a, §2.1]. In particular, as in [TV25a, Ex. 2.1.5.2], any map $f : X \to Y$ of derived Artin stacks induces a derived foliation on X, which intuitively corresponds to the decomposition of X into leaves which have constant value by f. Given a vector field on an affine X, by [TV25a, Ex. 2.1.5.5], we also obtain a derived foliation on X whose leaves morally correspond to paths that integrate this vector field (and this phenomenon is expected to extend to derived Artin stacks). Finally, given an action of a smooth scheme group G on a derived Artin stack X, by [TV25a, Ex. 2.1.5.6] we can also construct a derived foliation on X which morally partitions X into its orbits under this action. These examples show that derived foliations arise naturally in many familiar geometric contexts.

Finally, derived foliations should also interact naturally with shifted Poisson and symplectic geometry, as it is conjectured that a shifted Poisson structure should yield a derived foliation endowed with a shifted symplectic structure along its leaves, mirroring the classical relation between Poisson manifolds and their symplectic leaf decompositions.

4.2 ALGEBRAIC DEFINITIONS

In this chapter, we will recall the definitions of derived foliations in derived algebraic geometry. In a first section, we will recall the algebraic definitions of graded mixed algebras, some constructions on them, the properties used to define derived foliations, and discuss their stability under the preceding constructions. In a second section, we will recall results from [Mou21] and [MRT22] to reformulate these algebraic definitions in a more geometric context, i.e. in terms of quasi-coherent sheaves over some derived stacks. We will first recall the definition of the structure of a graded mixed algebra over the base field k, which we will call the absolute definition. Graded mixed algebras are essentially graded algebras equipped with internal differentials shifting the grading, whose main class of examples is de Rham algebras. Secondly, we will recall the definitions of graded and graded mixed objects defined over other k-algebras, which we will call the relative definitions.

All the definitions in this section can be found in [PTVV] and [TV25a].

4.2.1 Absolute Definitions

Notation 4.2.1. Let us list here the notations we use related to graded objects:

- The 1-category of $\mathbb{Z}$-graded k-dg modules, which are families of k-dg-modules $(E(n))_{n\in\mathbb{Z}}$, is defined as:

$$dg^{gr}_{st} := \mathrm{Fun}(\mathbb{Z}^{dis}, dg_k)$$

As in [PTVV], we use the cohomological conventions, so its objects and all complexes in this thesis are to be considered as cochain complexes.

- We will refer to its objects as graded modules, and the extra grading will be called the weight grading.
- The 1-category dg^{gr}_{st} is canonically equipped with a class of weak equivalences comprised of graded maps that induce quasi-isomorphism of k-dg-modules in each weight.
- The 1-category dg^{gr}_{st} also has a canonical symmetric monoidal structure, given by a Cauchy product formula, and the 1-category of commutative algebras is denoted by:

$$cdga^{gr}_{st} := CAlg(dg^{gr}_{st})$$

- The 1-category $cdga^{gr}_{st}$ inherits weak equivalences from dg^{gr}_{st}.

Definition 4.2.2. *Given a graded module E, a mixed structure on E consists in a family of maps of dg-modules $\epsilon_n : E(n) \to E(n+1)[-1]$ (where $[-1]$ denotes shifting the cohomological grading of the complex by -1), in particular satisfying $d \circ \epsilon + \epsilon \circ d = 0$, and moreover satisfying $\epsilon^2 = 0$.*

This data can be visualized in a diagram of the following shape, in which each column is a given weight, downwards arrows are homological differentials within each weight, and diagonal arrows are the mixed differentials:

$$\begin{tikzcd}
\ddots \arrow[dr, "\epsilon_{-2,-2}"] & \vdots \arrow[d, "d_{-1,-2}"] \arrow[dr, "\epsilon_{-1,-2}"] & \vdots \arrow[d, "d_{0,-2}"] \arrow[dr, "\epsilon_{0,-2}"] & \vdots \arrow[d, "d_{1,-2}"] & \iddots \\
\cdots \arrow[dr, "\epsilon_{-2,-1}"] & E(-1)_{-1} \arrow[d, "d_{-1,-1}"] \arrow[dr, "\epsilon_{-1,-1}"] & E(0)_{-1} \arrow[d, "d_{0,-1}"] \arrow[dr, "\epsilon_{0,-1}"] & E(1)_{-1} \arrow[d, "d_{1,-1}"] \arrow[dr, "\epsilon_{1,-1}"] & \cdots \\
\cdots \arrow[dr, "\epsilon_{-2,0}"] & E(-1)_0 \arrow[d, "d_{-1,0}"] \arrow[dr, "\epsilon_{-1,0}"] & E(0)_0 \arrow[d, "d_{0,0}"] \arrow[dr, "\epsilon_{0,0}"] & E(1)_0 \arrow[d, "d_{1,0}"] \arrow[dr, "\epsilon_{1,0}"] & \cdots \\
\cdots & E(-1)_1 \arrow[d, "d_{-1,1}"] \arrow[dr, "\epsilon_{-1,-1}"] & E(0)_1 \arrow[d, "d_{0,1}"] \arrow[dr, "\epsilon_{0,-1}"] & E(1)_1 \arrow[d, "d_{1,1}"] \arrow[dr, "\epsilon_{1,1}"] & \cdots \\
\iddots & \vdots & \vdots & \vdots & \ddots
\end{tikzcd}$$

Notation 4.2.3. Let us list here the notations we use related to graded mixed objects:

- A graded module equipped with a mixed structure is called a graded mixed module.
- Morphisms of graded mixed modules are morphisms of graded modules commuting with the differentials.
- The 1-category of graded mixed modules and their morphisms is denoted by:

$$\epsilon - dg^{gr}_{st}$$

- The 1-category $\epsilon - dg^{gr}_{st}$ is canonically equipped with a class of weak equivalences comprised of morphisms that induce quasi-isomorphisms in each weight.
- This 1-category can also be canonically equipped with a symmetric monoidal structure similar to that of dg^{gr}_{st}.
- A commutative algebra in this monoidal category is called a graded mixed algebra, and their 1-category will be denoted by:

$$\epsilon - cdga^{gr}_{st} := CAlg(\epsilon - dg^{gr}_{st})$$

  Note that in such algebras, the mixed differential ϵ will moreover satisfy Leibniz' rule, as a consequence of the definition of the monoidal structure of $\epsilon - dg^{gr}_{st}$.
- The 1-category $\epsilon - cdga^{gr}_{st}$ also comes equipped with a class of weak equivalences inherited from $\epsilon - dg^{gr}_{st}$.

Remark 4.2.4. The 1-categories dg^{gr}_{st}, $cdga^{gr}_{st}$, $\epsilon - dg^{gr}_{st}$, and $\epsilon - cdga^{gr}_{st}$ all come equipped with canonical classes of weak equivalences, as described above. As in [PTVV, below Prop. 1.3], we can localize these categories by their classes of weak equivalences, which yields ∞-categories.

Notation 4.2.5. We will call dg^{gr}, $cdga^{gr}$, $\epsilon - dg^{gr}$, and $\epsilon - cdga^{gr}$ the ∞-categories obtained after localization of the corresponding above 1-categories by their classes of weak equivalences.

4.2.2 Relative Definitions

We will now introduce definitions relative to a different base k-algebra than solely k. Note that in [PTVV], only the relative case over k is considered, but this generalization is mild.

Definition 4.2.6. *Let B be a connective k-cdga, and equip it with the trivial grading, i.e. where B is concentrated in weight 0. We can thus canonically consider that $B \in CAlg(dg^{gr}) = cdga^{gr}$. We will denote by $dg^{gr}(B)$ (resp. $cdga^{gr}(B)$) the ∞-category of B-modules (resp. B-algebras) in dg^{gr}, i.e.:*

$$dg^{gr}(B) := B - Mod(dg^{gr})$$

$$cdga^{gr}(B) := B - CAlg(dg^{gr})$$

Remark 4.2.7. Note that the ∞-category $cdga^{gr}(B)$ is canonically equivalent to the slice ∞-category $B/cdga^{gr}$.

This defines graded objects over B. Similarly we can define relative graded mixed objects. For this let us first consider the main example of graded mixed algebras.

Example 4.2.8. Let $\phi : A \to B$ be any homomorphism of connective k-cdga's, making B into an A-algebra. Recall that $\mathbb{L}_{B/A} \in B - Mod$ denotes the relative cotangent module of ϕ, and consider the relative de Rham algebra:

$$\mathbf{DR}(B/A) := \mathrm{Sym}_B(\mathbb{L}_{B/A}[1])$$

It is canonically equipped with the structure of a graded mixed algebra.

- The application of Sym[1] provides an extra grading, its weight grading, and a commutative algebra structure compatible with this grading. This constitutes its graded cdga structure.
- The de Rham differential satisfies the conditions to be a mixed structure on this graded cdga.

Therefore we can canonically consider that:

$$\mathbf{DR}(B/A) \in CAlg(\epsilon - dg^{gr}) = \epsilon - cdga^{gr}$$

This allows us to define relative graded mixed algebras in the following way.

Definition 4.2.9. *The ∞-category of A-linear graded mixed B-modules is defined as the ∞-category of $\mathbf{DR}(B/A)$-modules in $\epsilon - dg^{gr}$, and is denoted by $\epsilon - dg^{gr}(B/A)$, i.e.:*

$$\epsilon - dg^{gr}(B/A) := \mathbf{DR}(B/A) - Mod(\epsilon - dg^{gr})$$

The ∞-category of A-linear graded mixed B-algebras is defined as the slice ∞-category $\mathbf{DR}(B/A)/\epsilon - cdga^{gr}$, and is denoted by $\epsilon - cdga^{gr}(B/A)$, i.e.:

$$\begin{aligned} \epsilon - cdga^{gr}(B/A) :=& \mathbf{DR}(B/A) - CAlg(\epsilon - dg^{gr}) \\ \simeq& \mathbf{DR}(B/A)/\epsilon - cdga^{gr} \end{aligned}$$

Remark 4.2.10. Note that by considering a graded mixed module or algebra F over $\mathbf{DR}(B/A)$, since B is the weight-0 part of the latter, we can induce an actual B-module/algebra structure on F, but also an A-module/algebra structure inherited from

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[1] In this context, we are working in characteristic zero, so contrary to the case of positive characteristics, we do not need to dinstinguish between the permutation-invariant and the divided power algebras.

the structural map $\phi : A \to B$. Moreover, since the structural action of $\mathbf{DR}(B/A)$ on F has to be compatible with the graded mixed structures, and since the mixed differential in $\mathbf{DR}(B/A)$ cancels on A, this requires that the mixed differential in F also cancels on A, i.e. that it is A-linear. This provides the justification for the name of the ∞-category, and also describes the forgetful functors $\epsilon - dg^{gr}(B/A) \to dg^{gr}(B)$ and $\epsilon - cdga^{gr}(B/A) \to cdga^{gr}(B)$.

Definition 4.2.11. *Let us define the two forgetful functors from graded mixed objects to graded objects:*

- *The forgetful functor which maps graded mixed modules to their underlying graded modules will be denoted by:*

$$-^{gr} : \epsilon - dg^{gr}(B/A) \to dg^{gr}(B)$$

- *By abuse of notation, the analogous functor on graded mixed algebras will also be denoted by:*

$$-^{gr} : \epsilon - cdga^{gr}(B/A) \to cdga^{gr}(B)$$

In fact we can break down these forgetful functors in a composition of two forgetful steps.

Notation 4.2.12. Let us set the following notations:

- The ∞-category of graded modules over $\mathbf{DR}(B/A)$ viewed as a graded algebra (without taking into account its de Rham differential) is:

$$\mathbf{DR}(B/A) - Mod(dg^{gr})$$

- The functor forgetting the mixed differential of a graded mixed module is:

$$\not\epsilon : \epsilon - dg^{gr}(B/A) \to \mathbf{DR}(B/A) - Mod(dg^{gr})$$

- The functor inducing a B-action on a graded $\mathbf{DR}(B/A)$-module along the structural map $B \to \mathbf{DR}(B/A)$ is:

$$-^{B} : \mathbf{DR}(B/A) - Mod(dg^{gr}) \to dg^{gr}(B)$$

Remark 4.2.13. The forgetful functor $-^{gr} : \epsilon - dg^{gr}(B/A) \to dg^{gr}(B)$ is the composite of the forgetful functors $\not\epsilon : \epsilon - dg^{gr}(B/A) \to \mathbf{DR}(B/A) - Mod(dg^{gr})$ and $-^{B} : \mathbf{DR}(B/A) - Mod(dg^{gr}) \to dg^{gr}(B)$. To summarize, so far we have defined the following ∞-categories and forgetful functors:

$$\begin{tikzcd}
\epsilon - dg^{gr}(B/A) \arrow[r, "\not\epsilon"] \arrow[rr, bend right, "-^{gr}"'] & \mathbf{DR}(B/A) - Mod(dg^{gr}) \arrow[r, "-^{B}"] & dg^{gr}(B)
\end{tikzcd}$$

Note that all of these functors are conservative.

4.2.3 Change of Base

We will now inspect what happens when we change our relative base B/A.

Construction 4.2.14. Consider a commutative square of connective k-cdga's:

$$\begin{tikzcd} B \arrow[r, "b"] & B' \\ A \arrow[u, "f"] \arrow[r, "a"'] & A' \arrow[u, "f'"'] \end{tikzcd}$$

Note that this data can equivalently be phrased as a map $\phi : f \to f'$ in Arr(dRings), where in this case ϕ is the data of a, b and the commutativity of the square. In this situation, ϕ induces canonical morphisms $\mathbf{DR}(B/A) \xrightarrow{\beta} b^*\mathbf{DR}(B/A) \xrightarrow{\theta} \mathbf{DR}(B'/A')$ of graded mixed algebras. We will denote by $\psi : \mathbf{DR}(B/A) \to \mathbf{DR}(B'/A')$ the composite $\theta \circ \beta$. We can use it to construct a push-forward/pull-back adjunction between $\epsilon - cdga^{gr}(B'/A')$ and $\epsilon - cdga^{gr}(B/A)$, mildly generalizing [TV23a, Prop. 1.2.3]:

- The push-forward functor $\phi_* : \epsilon - cdga^{gr}(B'/A') \to \epsilon - cdga^{gr}(B/A)$ takes a graded mixed algebra F equipped with a structural map $\alpha' : \mathbf{DR}(B'/A') \to F$ and associates to it the composite $\mathbf{DR}(B/A) \xrightarrow{\psi} \mathbf{DR}(B'/A') \xrightarrow{\alpha'} F$.
- The pull-back functor $\phi^* : \epsilon - cdga^{gr}(B/A) \to \epsilon - cdga^{gr}(B'/A')$ takes a graded mixed algebra F equipped with a structural map $\alpha : \mathbf{DR}(B/A) \to F$, and associates to it the graded mixed algebra $\phi^*F := b^*F \overset{\mathbb{L}}{\otimes}_{b^*\mathbf{DR}(B/A)} \mathbf{DR}(B'/A')$, equipped with its structural map $\mathbf{DR}(B'/A') \to \phi^*F$, or more explicitly it is the map labeled $\phi^*\alpha$ in the following pushout in graded mixed algebras

$$\begin{tikzcd} b^*\mathbf{DR}(B/A) \arrow[r, "b^*\alpha"] \arrow[d, "\theta"'] & b^*F \arrow[d] \\ \mathbf{DR}(B'/A') \arrow[r, "\phi^*\alpha"'] & \phi^*F \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

- The two functors above form an adjunction $\phi^* \dashv \phi_*$.

Some A-linear graded mixed B-algebras serve as models for derived foliations on Spec(B) relative to Spec(A). We will now discuss the properties defining them.

Definition 4.2.15. *An A-linear graded mixed B-algebra F will be called* quasi-free *if the natural map* $\mathrm{Sym}_{F(0)} F(1) \to F$ *is an equivalence of graded algebras.*

We can show that this property is stable by push-forward.

Lemma 4.2.16. *Let $F' \in \epsilon - cdga^{gr}(B'/A')$, and let $F = \phi_*(F') \in \epsilon - cdga^{gr}(B/A)$ denote its push-forward. If F' is quasi-free, then so is F.*

Proof. We have to show that the natural map $\mathrm{Sym}_{F(0)} F(1) \to F$ is an equivalence, assuming $\mathrm{Sym}_{F'(0)} F'(1) \to F'$ already is. But F and F' are canonically isomorphic as graded algebras (forgetting the respective B and B'-algebra structures), so in particular $F(0)$ and $F'(0)$ are canonically equivalent as algebras, and $F(1)$ and $F'(1)$ also are canonically equivalent as $F(0)$-modules. Therefore, we can construct this commutative square of graded algebras:

$$\begin{tikzcd}
\mathrm{Sym}_{F(0)} F(1) \arrow[r] \arrow[d, "\sim"'] & F \arrow[d, "\sim"] \\
\mathrm{Sym}_{F'(0)} F'(1) \arrow[r, "\sim"] & F'
\end{tikzcd}$$

and thus the top map is also an equivalence of graded algebras. □

To conclude this section, we give the definition of a derived foliation over $\mathrm{Spec}(B)$ relative to $\mathrm{Spec}(A)$.

Definition 4.2.17. *An A-linear graded mixed B-algebra F will be called a* derived foliation on B relative to A *if it satisfies the following conditions:*

1. *The structural map $B \to F(0)$ is an equivalence*
2. *F is quasi-free*
3. *$F(1)$ is perfect over B*

We define the ∞-category $\mathcal{F}ol(B/A) \subset \epsilon - cdga^{gr}(B/A)^{\mathrm{op}}$ as the full subcategory spanned by these objects (note the op*). Moreover, the B-module $F(1)[-1]$ will be denoted by $\mathbb{L}_F$ and is called the cotangent complex of F which is required to be perfect by condition 3, while condition 1. and 2. are equivalent to saying that $F \simeq \mathrm{Sym}_B(\mathbb{L}_F[1])$ canonically as graded B-algebras.*

Remark 4.2.18. Note that more generally we can define non-perfect derived foliations by removing condition 3, which is the point of view taken in [TV25a, Def. 2.1.1.2, 2.1.1.3]. However, in this thesis, all derived foliations will be assumed to be perfect, with the sole exception of the next remark, in which we will explicitly mention that we are considering non-perfect derived foliations to emphasize that we remove condition 3.

A conceptual motivation for requiring condition 3, i.e. that the cotangent of the foliation is perfect, is that we wish to view foliations as a way to encode sub-bundles of the tangent bundle of the ambient space (equipped with a Lie algebra structure). Therefore, we need to be able to dualize the cotangent to obtain this tangent structure. However, we do not claim that condition 3. is the most general condition under which a notion of dual of the cotangent might exist, as for instance requiring instead that the cotangent is almost-perfect also yields a notion of dual, but only as a Pro-object in this case. Nonetheless, condition 3. is general enough for the examples of this thesis, and more general conditions will not be studied here.

Remark 4.2.19. Firstly, conditions 1. and 2. of definition 4.2.17 are stable by the pull-back of graded mixed algebras ϕ^* defined in construction 4.2.14, so that the pull-back of graded mixed algebras restricts to a pull-back of non-perfect derived foliations, as described in the non-relative case in [TV25a, Def. 2.1.3], and as we will show explicitly below.

Secondly, note that condition 1. in definition 4.2.17 is not stable by the push-forward of graded mixed algebras ϕ_* defined in construction 4.2.14, since it does not change the underlying graded mixed algebra and quasi-free B'-algebras need not remain quasi-free when viewed as B-algebras. Therefore, in general we cannot construct a push-forward of derived foliations naively as a restriction of the push-forward of graded mixed algebras. Instead, we will construct a push-forward of (perfect) derived foliations in chapter 6 using a different approach, through a point of view which we will introduce in the next section and develop in chapter 5. Moreover, note that all the conditions of definition 4.2.17 only depend on the graded structure of F, and not on its mixed structure.

Coming back to our first point, we will now show explicitly that conditions 1. and 2. in definition 4.2.17 are stable by the pull-back as defined in construction 4.2.14. Consider a commutative square

$$\begin{tikzcd}
B \arrow[r, "b"] & B' \\
A \arrow[u, "f"] \arrow[r, "a"'] & A' \arrow[u, "f'"']
\end{tikzcd}$$

and denote by $\phi : f \to f'$ the corresponding map in $\mathrm{Arr}(\mathrm{dRings})$. Then ϕ^* has the following description on non-perfect derived foliations: for $F \in \epsilon - cdga^{gr}(B/A)$, by definition, there is a pushout square in graded mixed algebras

$$\begin{tikzcd}
b^*\mathbf{DR}(B/A) \arrow[r, "b^*\alpha"] \arrow[d, "\theta"'] & b^*F \arrow[d] \\
\mathbf{DR}(B'/A') \arrow[r, "\phi^*\alpha"'] \arrow[r, phantom, "\ulcorner", very near end, shift left=3] & \phi^*F
\end{tikzcd}$$

If $F \simeq \mathrm{Sym}_B(\mathbb{L}_F[1])$ is a non-perfect derived foliation, we can identify $b^*F \simeq \mathrm{Sym}_{B'}(b^*\mathbb{L}_{\mathcal{F}}[1])$, and moreover $b^*\mathbf{DR}(B/A) \simeq \mathrm{Sym}_{B'}(b^*\mathbb{L}_{B/A}[1])$, so we can rewrite this pushout as

$$\begin{tikzcd}
\mathrm{Sym}_{B'}(b^*\mathbb{L}_{B/A}[1]) \arrow[r, "b^*\alpha"] \arrow[d, "\theta"'] & \mathrm{Sym}_{B'}(b^*\mathbb{L}_{\mathcal{F}}[1]) \arrow[d] \\
\mathrm{Sym}_{B'}(\mathbb{L}_{B'/A'}[1]) \arrow[r, "\phi^*\alpha"'] \arrow[r, phantom, "\ulcorner", very near end, shift left=3] & \phi^*F
\end{tikzcd}$$

Since $\mathrm{Sym}_{B'}$ is a left adjoint, it commutes with colimits, and thus we obtain that $\phi^*F \simeq \mathrm{Sym}_{B'}(\mathbb{L}_{\phi^*\mathcal{F}}[1])$, where $\mathbb{L}_{\phi^*\mathcal{F}}$ is the following pushout of B'-modules

$$\begin{tikzcd}
b^*\mathbb{L}_{B/A} \arrow[r] \arrow[d] & b^*\mathbb{L}_{\mathcal{F}} \arrow[d] \\
\mathbb{L}_{B'/A'} \arrow[r] & \mathbb{L}_{\phi^*\mathcal{F}} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

As a result, we obtain that ϕ^*F is a non-perfect derived foliation on B' relative to A', and that we have the following formula to compute its cotangent:

$$\mathbb{L}_{\phi^*\mathcal{F}} \simeq b^*\mathbb{L}_{\mathcal{F}} \oplus_{b^*\mathbb{L}_{B/A}} \mathbb{L}_{B'/A'}$$

In general, $\mathbb{L}_{\phi^*\mathcal{F}}$ may fail to remain perfect even if $\mathbb{L}_{\mathcal{F}}$ was perfect. However, note that if $\mathbb{L}_{B/A}$ and $\mathbb{L}_{B'/A'}$ are also perfect, e.g. if f and f' are of finite presentation, then $b^*\mathbb{L}_{B/A}$ is also perfect because b^* preserves perfects, and finally $\mathbb{L}_{\phi^*\mathcal{F}}$ is perfect since perfect B'-modules are stable by pushouts. Moreover, even without these assumptions, $\mathbb{L}_{\phi^*\mathcal{F}}$ may happen to be perfect for other reasons, as will be the case in the proof of proposition 6.2.5.

4.3 A Geometric Reformulation

We will now review results from [Mou21] (resp. [MRT22]), which provide equivalences between graded modules (resp. graded mixed modules) over a derived stack X and quasi-coherent sheaves on derived stacks associated to X. In the next section, this will allow us to provide equivalences between bundles over these stacks and graded algebras (resp. graded mixed algebras) satisfying extra regularity conditions, such as derived foliations.

More precisely, in the previous section we have introduced the following diagram (see remark 4.2.13):

$$\begin{tikzcd}
\epsilon - dg^{gr}(B/A) \arrow[r, "\not\epsilon"] \arrow[rr, bend right=0, swap, "\_gr", shift right=2] & \mathbf{DR}(B/A) - Mod(dg^{gr}) \arrow[r, "\_B"] & dg^{gr}(B)
\end{tikzcd}$$

In this section, our goal is to view each ∞-category of this diagram as the ∞-category of quasi-coherent sheaves over a given stack, and each functor between them as either a pull-back or a push-forward along a given map between those stacks.

Remark 4.3.1. Consider first absolute graded modules and algebras. As is well known in algebraic geometry, and has been shown in derived geometry in [Mou21, Thm. 4.1], there is a canonical equivalence of symmetric monoidal ∞-categories $dg^{gr} \simeq \mathrm{QCoh}(B\mathbb{G}_m)$, which promotes to an equivalence between their ∞-categories of commutative algebras. Moreover, by [HA, Thm. 4.5.4.7], the ∞-category of ∞-commutative algebras in the ∞-categorical localization of dg^{gr}_{st} is equivalent to the ∞-categorical localization of $cdga^{gr}_{st} = \mathrm{CAlg}(dg^{gr}_{st})$, the latter of which is by definition the ∞-category of graded mixed algebras. In other words, if $\mathcal{W}$ (resp. $\mathcal{W}'$) is the class of weak equivalences of dg^{gr}_{st} (resp. $cdga^{gr}_{st}$), then

$\infty - \mathrm{CAlg}(dg^{gr}_{st}[\mathcal{W}^{-1}]) \simeq \mathrm{CAlg}(dg^{gr}_{st})[\mathcal{W}'^{-1}]$. In conclusion, we have canonical equivalences of symmetric monoidal ∞-categories:

$$\begin{aligned} dg^{gr} &\simeq \mathrm{QCoh}(B\mathbb{G}_m) \\ cdga^{gr} &\simeq \mathrm{CAlg}(\mathrm{QCoh}(B\mathbb{G}_m)) \end{aligned}$$

To obtain a similar equivalence for graded mixed objects, let us now briefly introduce the group scheme called $\mathcal{H}$, by a definition which is only valid since we are working in characteristic zero, as introduced in [PTVV, Rmk. 1.5].[2]

Definition 4.3.2. *Let $\mathbb{G}_m$ and $\mathbb{G}_a$ denote respectively the multiplicative and additive group schemes. Recall that there is a canonical action of $\mathbb{G}_m$ on $\mathbb{G}_a$ by multiplication, which induces a canonical action of $\mathbb{G}_m$ on $B\mathbb{G}_a$ after delooping. Note that in derived algebraic geometry, all these group schemes including $B\mathbb{G}_a$ are affine and of finite type over k. We will call $\mathcal{H}$ the semi-direct product $B\mathbb{G}_a \rtimes \mathbb{G}_m$, i.e. there is a canonical split short exact sequence of abelian group schemes:*

$$1 \to B\mathbb{G}_a \to \mathcal{H} \leftrightarrows \mathbb{G}_m \to 1$$

Moreover, note that $\mathcal{H}$ is thus affine and of finite type over k too.

Remark 4.3.3. Consider now absolute graded mixed modules and algebras. In this context, $\mathcal{H}$-actions are equivalent to graded mixed objects. More precisely, $\mathcal{H}$ is closely related to the object $(S^1_{Fil})^{gr}$ defined over $B\mathbb{G}_m$ in [MRT22, Def. 1.2.2, Def. 2.2.11], in the sense that the usual delooping $B\mathcal{H}$ is equivalent to $B_{B\mathbb{G}_m}((S^1_{Fil})^{gr})$ the relative delooping of $(S^1_{Fil})^{gr}$ in the ∞-topos $\mathrm{dSt}/B\mathbb{G}_m$ (see [Rob24, Rmk. 2.3.65]). Thus, by collecting the results [MRT22, Thm. 1.2.1(ii), Thm. 1.2.1(iii), Prop. 4.2.3(ii)] there is a canonical equivalence of monoidal ∞-categories $\epsilon - dg^{gr} \simeq \mathrm{QCoh}(B\mathcal{H})$. Note that this equivalence has also been proven directly in characteristic zero in [PT22, Prop. 2.7]. In conclusion, by a similar reasoning as for graded objects, we obtain the following equivalences of symmetric monoidal ∞-categories:

$$\begin{aligned} \epsilon - dg^{gr} &\simeq \mathrm{QCoh}(B\mathcal{H}) \\ \epsilon - cdga^{gr} &\simeq \mathrm{CAlg}(\mathrm{QCoh}(B\mathcal{H})) \end{aligned}$$

We will now refine these equivalences to the relative cases, i.e. $cdga^{gr}(B)$ and $\epsilon - cdga^{gr}(B/A)$.

---

[2] In positive characteristic, two groups $\mathcal{H}$ and $\mathcal{H}_\pi$ exist to encode respectively derived foliations and infinitesimal derived foliations, see [TV23b, §1.2]. The pull-back of these two groups over $\mathbb{Q}$ both coincide with the group $\mathcal{H}$ as defined above, so the distinction disappears in our setting. We expect constructions similar to those of this thesis to also apply for these two groups.

Remark 4.3.4. For the graded case, consider first the trivial $\mathbb{G}_m$-action on Spec(B): its quotient is $[\mathrm{Spec}(B)/\mathbb{G}_m] = \mathrm{Spec}(B) \times B\mathbb{G}_m$, and similarly as before we have symmetric monoidal equivalences:

$$\begin{aligned} dg^{gr}(B) &\simeq \mathrm{QCoh}(\mathrm{Spec}(B) \times B\mathbb{G}_m) \\ cdga^{gr}(B) &\simeq \mathrm{CAlg}(\mathrm{QCoh}(\mathrm{Spec}(B) \times B\mathbb{G}_m)) \end{aligned}$$

For the graded mixed case, let us first introduce an auxiliary derived stack.

Definition 4.3.5. *For any derived stack Z over a derived stack X, we will call graded loops on Z relative to X the following mapping stack, named after the object $(L_{Fil}X)^{gr}$ in [MRT22, Thm. 5.1.3(2)]:*

$$\mathcal{L}^{gr}(Z/X) = \underline{\mathbf{Map}}_X(B\mathbb{G}_a \times X, Z)$$

It comes equipped with a canonical $\mathcal{H}$-action by acting on the source (more precisely, $B\mathbb{G}_a$ acts on the source, and $\mathbb{G}_m$ acts on $B\mathbb{G}_a$, which induces the $\mathcal{H}$-action), and a canonical map $\mathcal{L}^{gr}(Z/X) \to Z$ by evaluation at the base point of $B\mathbb{G}_a \times X$ over X. Moreover, when both Z and X are equipped with the trivial $\mathcal{H}$-actions, we have that:

1. *The induced map $\mathcal{L}^{gr}(Z/X) \to X$ is $\mathcal{H}$-equivariant.*
2. *The canonical map $\mathcal{L}^{gr}(Z/X) \to Z$ is not $\mathcal{H}$-equivariant, but it is $\mathbb{G}_m$-equivariant for the induced actions.*

Moreover, when $Z = \mathrm{Spec}(B)$ and $X = \mathrm{Spec}(A)$ are affines, by [MRT22, Thm. 5.1.3(2)], applied over the base ring A, which is also in characteristic zero (i.e. a $\mathbb{Q}$-algebra) since k is, this last stack is equivalent to $\mathrm{Spec}_B(\mathbf{DR}(B/A))$, where $\mathbf{DR}(B/A)$ is the (graded mixed) B-algebra introduced in example 4.2.8.

Remark 4.3.6. Note that since $\mathcal{L}^{gr}(B/A)$ is relatively affine over the affine Spec(B), it is relatively affine over the point $* = \mathrm{Spec}(k)$[3], and moreover the unique map to the point is also $\mathcal{H}$-equivariant since it was already the case over Spec(A), and $\mathrm{Spec}(A) \to *$ is $\mathcal{H}$-equivariant for the trivial actions. It follows that the quotient map $[\mathcal{L}^{gr}(B/A)/\mathcal{H}] \to B\mathcal{H}$ is relatively affine, and in fact $[\mathcal{L}^{gr}(B/A)/\mathcal{H}] \simeq \mathrm{Spec}_{B\mathcal{H}}(\mathbf{DR}(B/A))$ as derived stacks over $B\mathcal{H}$, where $\mathbf{DR}(B/A)$ is viewed as an object of CAlg(QCoh$(B\mathcal{H})$) under the equivalence of remark 4.3.3. Therefore, by [GR17, Chap. I.3, Prop. 3.3.3] there is a canonical symmetric monoidal equivalence:

$$\mathrm{QCoh}([\mathcal{L}^{gr}(B/A)/\mathcal{H}]) \simeq \mathbf{DR}(B/A) - \mathrm{Mod}(\mathrm{QCoh}(B\mathcal{H}))$$

---

[3] More concretely $\mathcal{L}^{gr}(B/A) \simeq \mathrm{Spec}_k(\mathbf{DR}(B/A))$ as derived stacks (over Spec(k)). In [Toë06, §2.2], this derived stack is simply denoted as Spec$(\mathbf{DR}(B/A))$. However, this notation can be confusing here since $\mathbf{DR}(B/A)$ is not a derived ring in the sense of this thesis, as it is not connective in general. Thus, the object Spec$(\mathbf{DR}(B/A))$ as defined in the above sense is not a derived affine in our sense.

which promotes to the following equivalences by remark 4.3.3 and definition 4.2.9:

$$\epsilon - dg^{gr}(B/A) \simeq \mathrm{QCoh}([\mathcal{L}^{gr}(B/A)/\mathcal{H}])$$
$$\epsilon - cdga^{gr}(B/A) \simeq \mathrm{CAlg}(\mathrm{QCoh}([\mathcal{L}^{gr}(B/A)/\mathcal{H}]))$$

Remark 4.3.7. We can give a similar geometric description to the forgetful functor $-^{gr} : \epsilon - dg^{gr}(B/A) \to dg^{gr}(B)$. On the algebraic side, recall by remark 4.2.13 that we have defined the following ∞-categories and forgetful functors:

$$\begin{tikzcd}
\epsilon - dg^{gr}(B/A) \arrow[r, "\not\epsilon"] \arrow[rr, bend right, "-^{gr}"'] & \mathbf{DR}(B/A) - Mod(dg^{gr}) \arrow[r, "-^{B}"] & dg^{gr}(B)
\end{tikzcd}$$

On the geometric side, note that the canonical map $\mathcal{L}^{gr}(B/A) \to \mathrm{Spec}(B)$ is $\mathbb{G}_m$-equivariant as stated in definition 4.3.5, and that $\mathbb{G}_m$ embeds as a subgroup of $\mathcal{H}$ by the splitting of the short exact sequence in definition 4.3.2, so we can form the following span of quotient stacks:

$$\begin{tikzcd}
{[\mathcal{L}^{gr}(B/A)/\mathcal{H}]} & {[\mathcal{L}^{gr}(B/A)/\mathbb{G}_m]} \arrow[l, "\pi"'] \arrow[r, "p"] & \mathrm{Spec}(B) \times \mathbb{G}_m
\end{tikzcd}$$

Recall that we also have the following equivalences:

- $dg^{gr}(B) \simeq \mathrm{QCoh}(\mathrm{Spec}(B) \times B\mathbb{G}_m)$ (see remark 4.3.4)
- $\epsilon - dg^{gr}(B/A) \simeq \mathrm{QCoh}([\mathcal{L}^{gr}(B/A)/\mathcal{H}])$ (see remark 4.3.6)
- $\mathbf{DR}(B/A) - Mod(dg^{gr}) \simeq \mathrm{QCoh}([\mathcal{L}^{gr}(B/A)/\mathbb{G}_m])$ (similarly)

Finally, note that under those equivalences, the functor $\not\epsilon$ corresponds to the pull-back $\pi^* : \mathrm{QCoh}([\mathcal{L}^{gr}(B/A)/\mathcal{H}]) \to \mathrm{QCoh}([\mathcal{L}^{gr}(B/A)/\mathbb{G}_m])$, and that the functor $-^{B}$ corresponds to the push-forward $p_* : \mathrm{QCoh}([\mathcal{L}^{gr}(B/A)/\mathbb{G}_m]) \to \mathrm{QCoh}(\mathrm{Spec}(B) \times \mathbb{G}_m)$.

To conclude this section, we can summarize it in the following way.

Conclusion 4.3.8. Starting from the following canonical span of quotient stacks:

$$\begin{tikzcd}
{[\mathcal{L}^{gr}(B/A)/\mathcal{H}]} & {[\mathcal{L}^{gr}(B/A)/\mathbb{G}_m]} \arrow[l, "\pi"'] \arrow[r, "p"] & \mathrm{Spec}(B) \times \mathbb{G}_m,
\end{tikzcd}$$

we can form the following commutative diagram:

$$\begin{tikzcd}
\mathrm{QCoh}([\mathcal{L}^{gr}(B/A)/\mathcal{H}]) \arrow[r, "\pi^*"] \arrow[d, phantom, "\wr"] & \mathrm{QCoh}([\mathcal{L}^{gr}(B/A)/\mathbb{G}_m]) \arrow[r, "p_*"] \arrow[d, phantom, "\wr"] & \mathrm{QCoh}(\mathrm{Spec}(B) \times B\mathbb{G}_m) \arrow[d, phantom, "\wr"] \\
\epsilon - dg^{gr}(B/A) \arrow[r, "\not\epsilon"] \arrow[rr, bend right, "-^{gr}"'] & \mathbf{DR}(B/A) - Mod(dg^{gr}) \arrow[r, "-^{B}"] & dg^{gr}(B)
\end{tikzcd}$$

Chapter 5

Derived Foliations as Equivariant Stacks

Compared to the previous chapter, in the present one we will let $Z = \mathrm{Spec}(B)$ be an affine derived stack over an affine base $X = \mathrm{Spec}(A)$, and replace all occurrences of B (resp. A) by Z (resp. X) in the notations of the previous chapter.

Our goal in this chapter is to describe the ∞-category of X-linear derived foliations over Z as a full subcategory of $\mathcal{H}$-equivariant derived stacks over $\mathcal{L}^{gr}(Z/X)$, the graded loop stack of Z over X. We will then globalize this result to the case where $Z \to X$ is a relative Deligne-Mumford stack. In order to prove it, we will follow a strategy similar to [Mon21]: we will construct an adjunction between X-linear graded-mixed algebras over Z and the latter ∞-category, show that it is fully faithful, and describe its essential image. In chapter 6, we will use the latter description of foliations to define their push-forward, as a restriction of the push-forward of stacks along a map $f : X \to Y$ satisfying extra conditions, after showing that it does preserve foliations.

5.1 Categorical Overview of the Setting

To make the goal of the present chapter more precise, this first section is dedicated to introducing all the ∞-categories of stacks, algebras of quasi-coherent sheaves, and all their relationships, half of which were introduced in chapter 4. As a result, this section is heavy in definitions and notations, the point of which is to arrive at the diagram presented at the end in conclusion 5.1.12.

To begin, we will use the conclusion of chapter 4.

Reminder 5.1.1. In conclusion 4.3.8, we arrived at the following setting.

- There is a canonical span of quotient stacks:

$$[\mathcal{L}^{gr}(Z/X)/\mathcal{H}] \xleftarrow{\pi} [\mathcal{L}^{gr}(Z/X)/\mathbb{G}_m] \xrightarrow{p} Z \times B\mathbb{G}_m$$

- The following diagram is commutative:

$$\begin{tikzcd}
\mathrm{QCoh}([\mathcal{L}^{gr}(Z/X)/\mathcal{H}]) \arrow[r, "\pi^*"] \arrow[d, phantom, "\rotatebox{90}{$\simeq$}"] & \mathrm{QCoh}([\mathcal{L}^{gr}(Z/X)/\mathbb{G}_m]) \arrow[r, "p_*"] \arrow[d, phantom, "\rotatebox{90}{$\simeq$}"] & \mathrm{QCoh}(Z \times B\mathbb{G}_m) \arrow[d, phantom, "\rotatebox{90}{$\simeq$}"] \\
\epsilon - dg^{gr}(Z/X) \arrow[r, "\notin"] \arrow[rr, bend right=0, "\_^{gr}"', shift right=3] & \mathbf{DR}(Z/X) - Mod(dg^{gr}) \arrow[r, "\_^{B}"] & dg^{gr}(Z)
\end{tikzcd}$$

This provides a description of the ∞-categories of quasi-coherent sheaves of those stacks in terms of the ∞-categories of graded and graded mixed modules introduced in section 4.2.

Remark 5.1.2. In parallel, we can inspect the ∞-categories of derived stacks over these stacks:

- By [HAGII, Prop. 1.3.5.1], the ∞-category $\mathrm{dSt}\,/[\mathcal{L}^{gr}(Z/X)/\mathcal{H}]$ is canonically equivalent to the ∞-category whose objects are derived stacks T over $\mathcal{L}^{gr}(Z/X)$, equipped with $\mathcal{H}$-actions, whose structural maps $T \to \mathcal{L}^{gr}(Z/X)$ are compatible with the actions of $\mathcal{H}$ on both sides, i.e. $\mathcal{H}$-equivariant, and maps between such objects $T \to T'$ are also required to be compatible with the $\mathcal{H}$-actions on both sides.
- The latter ∞-category is usually called the ∞-category of $\mathcal{H}$-equivariant derived stacks over $\mathcal{L}^{gr}(Z/X)$, and denoted by $\mathrm{dSt}\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H}$.
- More concretely, the equivalence $\mathrm{dSt}\,/[\mathcal{L}^{gr}(Z/X)/\mathcal{H}] \xrightarrow{\sim} \mathrm{dSt}\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H}$ of [HAGII, Prop. 1.3.5.1] is given by taking an object $T \to [\mathcal{L}^{gr}(Z/X)/\mathcal{H}]$, and pulling it back along the canonical map $\mathcal{L}^{gr}(Z/X) \to [\mathcal{L}^{gr}(Z/X)/\mathcal{H}]$. In the converse direction, it takes an $\mathcal{H}$-equivariant derived stack $T' \to \mathcal{L}^{gr}(Z/X)$, and associates to it the quotient the quotient map $[T'/\mathcal{H}] \to [\mathcal{L}^{gr}(Z/X)/\mathcal{H}]$.
- Similar remarks can be made about $\mathrm{dSt}\,/[\mathcal{L}^{gr}(Z/X)/\mathbb{G}_m]$ and $\mathrm{dSt}\,/(Z \times B\mathbb{G}_m)$.

In conclusion, we have:

$$\begin{aligned}
\mathrm{dSt}\,/[\mathcal{L}^{gr}(Z/X)/\mathcal{H}] &\simeq \mathrm{dSt}\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H} \\
\mathrm{dSt}\,/[\mathcal{L}^{gr}(Z/X)/\mathbb{G}_m] &\simeq \mathrm{dSt}\,/\mathcal{L}^{gr}(Z/X) - \mathbb{G}_m \\
\mathrm{dSt}\,/(Z \times B\mathbb{G}_m) &\simeq \mathrm{dSt}\,/Z - \mathbb{G}_m
\end{aligned}$$

Our current goal is to relate these ∞-categories with the ∞-categories of quasi-coherent sheaves described in reminder 5.1.1. To proceed, let us now also recall a standard duality between derived stacks defined over a base stack S and commutative algebras of quasi-coherent sheaves over S.

Reminder 5.1.3. Let S be any derived stack.

- Let $p : T \to S \in \mathrm{dSt}\,/S$ be a map of derived stacks. We define $\mathcal{O}_{/S}(p) = p_*(\mathcal{O}_T) \in \mathrm{CAlg}(\mathrm{QCoh}(S))$, where $\mathcal{O}_T$ is the structure sheaf of T, and where $p_* : \mathrm{CAlg}(\mathrm{QCoh}(T)) \to \mathrm{CAlg}(\mathrm{QCoh}(S))$ is the push-forward functor of quasi-coherent sheaves. This canonically promotes to a contravariant functor, called the S-relative structure sheaf, and denoted by:

$$\mathcal{O}_{/S} : \mathrm{dSt}\,/S \to \mathrm{CAlg}(\mathrm{QCoh}(S))^{\mathrm{op}}$$

- Conversely, let $A \in \mathrm{CAlg}(\mathrm{QCoh}(S))$ be a quasi-coherent algebra. We define $\mathrm{Spec}_S(A) \in \mathrm{dSt}\,/S$ to be the derived stack on S defined on the generating site $\mathrm{dAff}\,/S$ by mapping $q : \mathrm{Spec}(B) \to S$ to $\mathbf{Map}_{B-Alg}(q^*A, B)$. This canonically promotes to a contravariant functor, called the S-relative spectrum, and denoted by:

$$\mathrm{Spec}_S : \mathrm{CAlg}(\mathrm{QCoh}(S))^{\mathrm{op}} \to \mathrm{dSt}\,/S$$

- These form the canonical adjoint pair:

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{QCoh}(S))^{\mathrm{op}} \arrow[r, shift right=1, "\mathrm{Spec}_S"'] \arrow[r, phantom, "\perp"] & \mathrm{dSt}\,/S \arrow[l, shift right=1, "\mathcal{O}_{/S}"']
\end{tikzcd}$$

Moreover, we can show that this construction itself is functorial in S, in the sense of the following remark.

Reminder 5.1.4. Let $f : S' \to S$ be any map of derived stacks.

- There is a forgetful/pull-back/push-forward adjunction triplet $f_! \dashv f^* \dashv f_*$ between their ∞-categories of relative derived stacks:

$$\begin{tikzcd}
\mathrm{dSt}\,/S' \arrow[r, shift left=3, "f_!"] \arrow[r, shift right=3, "f_*"'] \arrow[r, phantom, shift left=1.5, "\perp"] \arrow[r, phantom, shift right=1.5, "\perp"] & \mathrm{dSt}\,/S \arrow[l, "f^*" description]
\end{tikzcd}$$

where, given $U \to S \in \mathrm{dSt}\,/S$, we define $f^*(U)$ to be its pull-back along f equipped with its canonical projection to S':

$$f^*(U) := U \times_S S' \to S' \in \mathrm{dSt}\,/S'$$

its left adjoint $f_! : \mathrm{dSt}\,/S' \to \mathrm{dSt}\,/S$ is given simply by composition along f, in the sense that if $U \to S'$ is an object of $\mathrm{dSt}\,/S'$, then we have:

$$f_!(U') := U \to S' \xrightarrow{f} S \in \mathrm{dSt}\,/S$$

Moreover, f^* also admits a right adjoint $f_* : \mathrm{dSt}\,/S' \to \mathrm{dSt}\,/S$, called the push-forward, direct image, or Weil restriction along f, given on an object $U' \to S'$ by the following functor of points:

$$\forall\, \mathrm{Spec}(A) \to S \in \mathrm{dAff}\,/S,\ f_*(U')(A) := \mathbf{Map}_{\mathrm{dSt}\,/S'}(\mathrm{Spec}(A) \times_S S', U')$$

More generally, this extends to arbitrary stacks over S, so $f_*(U')$ satisfies the following universal property in $\mathrm{dSt}\,/S$:

$$\forall T \to S,\ \mathbf{Map}_S(T, f_*(U')) \simeq \mathbf{Map}_{S'}(T \times_S S', U')$$

which recovers exactly the statement that f_* is the right adjoint of f^*.

- There is a pull-back/push-forward adjunction between their ∞-categories of algebras of quasi-coherent sheaves:

$$\begin{tikzcd} \mathrm{CAlg}(\mathrm{QCoh}(S'))^{\mathrm{op}} \arrow[d, shift right=1.5ex, "f_*"'] \arrow[d, phantom, "\dashv"] \\ \mathrm{CAlg}(\mathrm{QCoh}(S))^{\mathrm{op}} \arrow[u, shift right=1.5ex, "f^*"'] \end{tikzcd}$$

  where, given $A \in \mathrm{CAlg}(\mathrm{QCoh}(S))$, we define:

$$f^*(A) := A \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \in \mathrm{CAlg}(\mathrm{QCoh}(S'))$$

  Note that usually f^* is considered as the left-adjoint to f_*, but in this diagram it is denoted as the right-adjoint to f_* instead, because we are taking opposites of ∞-categories on both sides.

- The following diagrams commute:

$$\begin{tikzcd} \mathrm{CAlg}(\mathrm{QCoh}(S'))^{\mathrm{op}} \arrow[d, "f_*"'] & \mathrm{dSt}\,/S' \arrow[l, "\mathcal{O}_{/S'}"'] \arrow[d, "f_!"] \\ \mathrm{CAlg}(\mathrm{QCoh}(S))^{\mathrm{op}} & \mathrm{dSt}\,/S \arrow[l, "\mathcal{O}_{/S}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{CAlg}(\mathrm{QCoh}(S'))^{\mathrm{op}} \arrow[r, "\mathrm{Spec}_{S'}"'] & \mathrm{dSt}\,/S' \\ \mathrm{CAlg}(\mathrm{QCoh}(S))^{\mathrm{op}} \arrow[u, "f^*"'] \arrow[r, "\mathrm{Spec}_S"'] & \mathrm{dSt}\,/S \arrow[u, "f^*"'] \end{tikzcd}$$

Remark 5.1.5. In particular, we can apply these constructions to our span from conclusion 4.3.8, i.e. apply it to:

$$[\mathcal{L}^{gr}(Z/X)/\mathcal{H}] \xleftarrow{\pi} [\mathcal{L}^{gr}(Z/X)/\mathbb{G}_m] \xrightarrow{p} Z \times B\mathbb{G}_m$$

Moreover, we can describe the corresponding ∞-categories of quasi-coherent sheaves by reminder 5.1.1, and of derived stacks by remark 5.1.2. We thus obtain the following diagram, which a priori need not commute:

$$\begin{tikzcd}
{[\mathcal{L}^{gr}(Z/X)/\mathcal{H}]} & \epsilon - cdga^{gr}(Z/X)^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_{[\mathcal{L}^{gr}(Z/X)/\mathcal{H}]}"'] \arrow[r, phantom, "\perp"] \arrow[d, "\pi^*"] & \mathrm{dSt}/\mathcal{L}^{gr}(Z/X) - \mathcal{H} \arrow[l, shift right, "\mathcal{O}_{[\mathcal{L}^{gr}(Z/X)/\mathcal{H}]}"'] \arrow[d, "\pi^*"] \\
{[\mathcal{L}^{gr}(Z/X)/\mathbb{G}_m]} \arrow[u, "\pi"] \arrow[r, rightsquigarrow] \arrow[d, "p"] & cdga^{gr}(\mathbf{DR}(Z/X))^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_{[\mathcal{L}^{gr}(Z/X)/\mathbb{G}_m]}"'] \arrow[r, phantom, "\perp"] \arrow[d, "p_*"] & \mathrm{dSt}/\mathcal{L}^{gr}(Z/X) - \mathbb{G}_m \arrow[l, shift right, "\mathcal{O}_{[\mathcal{L}^{gr}(Z/X)/\mathbb{G}_m]}"'] \arrow[d, "p_!"] \\
{[Z/\mathbb{G}_m]} & cdga^{gr}(Z)^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_{[Z/\mathbb{G}_m]}"'] \arrow[r, phantom, "\perp"] & \mathrm{dSt}/Z - \mathbb{G}_m \arrow[l, shift right, "\mathcal{O}_{[Z/\mathbb{G}_m]}"']
\end{tikzcd}$$

Note that so far, by the last point of reminder 5.1.4, we only know that the following diagrams are commutative:

$$\begin{tikzcd}
\epsilon - cdga^{gr}(Z/X)^{\mathrm{op}} \arrow[r, "\mathrm{Spec}_{[\mathcal{L}^{gr}(Z/X)/\mathcal{H}]}"] \arrow[d, "\pi^*"'] & \mathrm{dSt}/\mathcal{L}^{gr}(Z/X) - \mathcal{H} \arrow[d, "\pi^*"] \\
cdga^{gr}(\mathbf{DR}(Z/X))^{\mathrm{op}} \arrow[r, "\mathrm{Spec}_{[\mathcal{L}^{gr}(Z/X)/\mathbb{G}_m]}"'] & \mathrm{dSt}/\mathcal{L}^{gr}(Z/X) - \mathbb{G}_m
\end{tikzcd}$$

$$\begin{tikzcd}
cdga^{gr}(\mathbf{DR}(Z/X))^{\mathrm{op}} \arrow[d, "p_*"'] & \mathrm{dSt}/\mathcal{L}^{gr}(Z/X) - \mathbb{G}_m \arrow[l, "\mathcal{O}_{[\mathcal{L}^{gr}(Z/X)/\mathbb{G}_m]}"'] \arrow[d, "p_!"] \\
cdga^{gr}(Z)^{\mathrm{op}} & \mathrm{dSt}/Z - \mathbb{G}_m \arrow[l, "\mathcal{O}_{[Z/\mathbb{G}_m]}"]
\end{tikzcd}$$

In the above diagrams, let us now give special notations to the functors involved.

Notation 5.1.6. For the functors obtained from reminder 5.1.3 which are of the form $\mathrm{dSt}/S : \mathcal{O}_{/S} \dashv \mathrm{Spec}_S : \mathrm{CAlg}(\mathrm{QCoh}(S))^{\mathrm{op}}$, we introduce notations in the following cases:

- When $S = [\mathcal{L}^{gr}(Z/X)/\mathcal{H}]$, we denote it by: $\mathcal{O}_{\epsilon-gr} \dashv \mathrm{Spec}^{\epsilon-gr}$.
- When $S = [\mathcal{L}^{gr}(Z/X)/\mathbb{G}_m]$, we denote it by: $\mathcal{O}_{\mathbf{DR}-gr} \dashv \mathrm{Spec}^{\mathbf{DR}-gr}$.
- When $S = [Z/\mathbb{G}_m]$, we denote it by: $\mathcal{O}_{gr} \dashv \mathrm{Spec}^{gr}$. Note that $\mathcal{O}_{gr}$ was introduced and studied in [Mon21, Def. 1.11].

Notation 5.1.7. For the functors obtained from reminder 5.1.4 which are of the form $\mathrm{CAlg}(\mathrm{QCoh}(S))^{\mathrm{op}} : f_* \dashv f^* : \mathrm{CAlg}(\mathrm{QCoh}(S'))^{\mathrm{op}}$ and $\mathrm{dSt}/S : f_! \dashv f^* : \mathrm{dSt}/S'$, we introduce notations in the following cases:

- When $f = \pi : [\mathcal{L}^{gr}(Z/X)/\mathcal{H}] \to [\mathcal{L}^{gr}(Z/X)/\mathbb{G}_m]$, we introduce:
  $\not\epsilon : \epsilon - cdga^{gr}(Z/X)^{\mathrm{op}} \to cdga^{gr}(\mathbf{DR}(Z/X))^{\mathrm{op}} := \pi^*$, following conclusion 4.3.8
  $-^{\mathbb{G}_m} : \mathrm{dSt}/\mathcal{L}^{gr}(Z/X) - \mathcal{H} \to \mathrm{dSt}/\mathcal{L}^{gr}(Z/X) - \mathbb{G}_m := \pi^*$

- When $f = p : [\mathcal{L}^{gr}(Z/X)/\mathbb{G}_m] \to [Z/\mathbb{G}_m]$, we introduce:
  $-^Z : cdga^{gr}(\mathbf{DR}(Z/X))^{\mathrm{op}} \to cdga^{gr}(Z)^{\mathrm{op}} := p_*$, following conclusion 4.3.8
  $-^Z : \mathrm{dSt}\,/\mathcal{L}^{gr}(Z/X) - \mathbb{G}_m \to \mathrm{dSt}\,/Z - \mathbb{G}_m := p_!$

Finally, we also introduce notations for the following compositions:
$-^{gr} : \epsilon - cdga^{gr}(Z/X)^{\mathrm{op}} \to cdga^{gr}(Z)^{\mathrm{op}} := -^Z \circ \not\epsilon$, following remark 4.2.13
$-^{Z-\mathbb{G}_m} : \mathrm{dSt}\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H} \to \mathrm{dSt}\,/Z - \mathbb{G}_m := -^Z \circ -^{\mathbb{G}_m}$

Remark 5.1.8. So far, we have the following diagram, which a priori need not commute:

$$\begin{tikzcd}
{} \arrow[dd, "-^{gr}"'] & \epsilon - cdga^{gr}(Z/X)^{\mathrm{op}} \arrow[r, phantom, "\perp"] \arrow[r, shift right=1.5ex, "\mathrm{Spec}^{\epsilon-gr}"'] \arrow[d, "\not\epsilon"'] & \mathrm{dSt}\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H} \arrow[l, shift right=1.5ex, "\mathcal{O}_{\epsilon-gr}"'] \arrow[d, "-^{\mathbb{G}_m}"] & {} \arrow[dd, "-^{Z-\mathbb{G}_m}"] \\
 & cdga^{gr}(\mathbf{DR}(Z/X))^{\mathrm{op}} \arrow[r, phantom, "\perp"] \arrow[r, shift right=1.5ex, "\mathrm{Spec}^{\mathbf{DR}-gr}"'] \arrow[d, "-^Z"'] & \mathrm{dSt}\,/\mathcal{L}^{gr}(Z/X) - \mathbb{G}_m \arrow[l, shift right=1.5ex, "\mathcal{O}_{\mathbf{DR}-gr}"'] \arrow[d, "-^Z"] & \\
{} & cdga^{gr}(Z)^{\mathrm{op}} \arrow[r, phantom, "\perp"] \arrow[r, shift right=1.5ex, "\mathrm{Spec}^{gr}"'] & \mathrm{dSt}\,/Z - \mathbb{G}_m \arrow[l, shift right=1.5ex, "\mathcal{O}_{gr}"'] & {}
\end{tikzcd}$$

As above, so far we know only of the following commutativity:

$$\mathrm{Spec}^{\mathbf{DR}-gr} \circ \not\epsilon \simeq -^{\mathbb{G}_m} \circ \mathrm{Spec}^{\epsilon-gr}$$
$$\mathcal{O}_{gr} \circ -^Z \simeq -^Z \circ \mathcal{O}_{\mathbf{DR}-gr}$$

To complete the above diagram, let us recall and introduce additional ∞-categories and functors involved in our setting.

Reminder 5.1.9. We recall following objects:

- The ∞-category $\mathcal{F}ol(Z/X) \subset \epsilon - cdga^{gr}(Z/X)^{\mathrm{op}}$ is the full subcategory spanned by graded mixed algebras satisfying the conditions of definition 4.2.17.
- We recall the following from [Mon21, Prop. 2.1]:
  - There is a forgetful functor:

$$-(1) : cdga^{gr}(Z)^{\mathrm{op}} \to \mathrm{QCoh}(Z)^{\mathrm{op}},$$

  which takes a graded Z-algebra and maps it to its part sitting in weight 1.
  - The functor $-(1)$ is left-adjoint to the functor:

$$\mathrm{Sym}_{\mathcal{O}_Z} : \mathrm{QCoh}(Z)^{\mathrm{op}} \to cdga^{gr}(Z)^{\mathrm{op}}$$

  - The composition $\mathrm{Spec}^{gr} \circ \mathrm{Sym}_{\mathcal{O}_Z}$ is denoted by $\mathbb{V} : \mathrm{QCoh}(Z)^{\mathrm{op}} \to \mathrm{dSt}\,/Z - \mathbb{G}_m$.
  - The essential image of $\mathbb{V}$ is called the ∞-category of linear stacks over Z.

Let us now introduce the following objects and notations.

Definition 5.1.10. *Let* $\mathrm{PerfLinSt}_Z \subset \mathrm{dSt}\,/Z - \mathbb{G}_m$ *be the* ∞*-category of perfect linear stacks over* Z*, defined as the essential image of* $\mathbb{V}_{|\,\mathrm{Perf}(Z)^{\mathrm{op}}} : \mathrm{Perf}(Z)^{\mathrm{op}} \to \mathrm{dSt}\,/Z - \mathbb{G}_m$*.*

Definition 5.1.11. *Let* $\mathrm{PerfLinSt}_Z\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H}$ *be defined as the full subcategory of* $\mathrm{dSt}\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H}$ *spanned by stacks whose image under the forgetful functor* $-^{Z-\mathbb{G}_m} : \mathrm{dSt}\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H} \to \mathrm{dSt}\,/Z - \mathbb{G}_m$ *belongs to* $\mathrm{PerfLinSt}_Z$*. Equivalently, we can define* $\mathrm{PerfLinSt}_Z\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H}$ *using the following pull-back square of* ∞*-categories:*

$$\begin{tikzcd}
\mathrm{PerfLinSt}_Z\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{PerfLinSt}_Z \arrow[d, hook] \\
\mathrm{dSt}\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H} \arrow[r, "\_^{Z-\mathbb{G}_m}"] & \mathrm{dSt}\,/Z - \mathbb{G}_m
\end{tikzcd}$$

To conclude this section, let us combine all its data.

Conclusion 5.1.12. We have constructed the following diagram, which a priori need not commute:

$$\begin{tikzcd}
 & \mathcal{F}ol(Z/X) \arrow[d, hook] & & \mathrm{PerfLinSt}_Z\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H} \arrow[d, hook] & \\
{} \arrow[dd, "\_^{gr}"] & \epsilon - cdga^{gr}(Z/X)^{\mathrm{op}} \arrow[d, "\not\epsilon"'] \arrow[rr, shift right, "\mathrm{Spec}^{\epsilon-gr}"'] \arrow[rr, phantom, "\perp"] & & \mathrm{dSt}\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H} \arrow[d, "\_^{\mathbb{G}_m}"] \arrow[ll, shift right, "\mathcal{O}_{\epsilon-gr}"'] & {} \arrow[dd, "\_^{Z-\mathbb{G}_m}"] \\
 & cdga^{gr}(\mathbf{DR}(Z/X))^{\mathrm{op}} \arrow[d, "\_^{Z}"'] \arrow[rr, shift right, "\mathrm{Spec}^{\mathbf{DR}-gr}"'] \arrow[rr, phantom, "\perp"] & & \mathrm{dSt}\,/\mathcal{L}^{gr}(Z/X) - \mathbb{G}_m \arrow[d, "\_^{Z}"] \arrow[ll, shift right, "\mathcal{O}_{\mathbf{DR}-gr}"'] & \\
{} & cdga^{gr}(Z)^{\mathrm{op}} \arrow[d, shift right, "-(1)"'] \arrow[d, phantom, "\dashv"] \arrow[rr, shift right, "\mathrm{Spec}^{gr}"'] \arrow[rr, phantom, "\perp"] & & \mathrm{dSt}\,/Z - \mathbb{G}_m \arrow[ll, shift right, "\mathcal{O}_{gr}"'] & {} \\
 & \mathrm{QCoh}(Z)^{\mathrm{op}} \arrow[u, shift right, "\mathrm{Sym}_{\mathcal{O}_Z}"'] \arrow[urr, "\mathbb{V}"'] & & &
\end{tikzcd}$$

In this diagram, so far we only know the following commutativity:

$$\mathrm{Spec}^{\mathbf{DR}-gr} \circ \not\epsilon \simeq -^{\mathbb{G}_m} \circ \mathrm{Spec}^{\epsilon-gr}$$

$$\mathcal{O}_{gr} \circ -^{Z} \simeq -^{Z} \circ \mathcal{O}_{\mathbf{DR}-gr}$$

$$\mathbb{V} := \mathrm{Spec}^{gr} \circ \mathrm{Sym}_{\mathcal{O}_Z}$$

By the end of the next section, in theorem 5.2.9, we will prove that the adjunction $\mathcal{O}_{\epsilon-gr} \dashv \mathrm{Spec}^{\epsilon-gr}$ restricts to an adjoint equivalence of ∞-categories:

$$\mathcal{F}ol(Z/X) \simeq \mathrm{PerfLinSt}_Z\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H}$$

To prove it, we will show that the vertical forgetful functors are compatible with the horizontal adjunctions, in a sense we will discuss in the next section. Essentially, we will prove that the diagram is in fact commutative. In the final section, section 5.3, we will globalize these results to non-affine cases, more specifically to the case of derived Deligne-Mumford stacks.

5.2 Equivalence in the Affine Case

As stated above, the equivalence in theorem 5.2.9 will follow from the fact that the squares of the diagram in conclusion 5.1.12 satisfy a compatibility condition. This condition is known as the Beck-Chevalley condition (note that squares satisfying this condition are also called adjointable squares in the terminology of [HA, Def. 4.7.4.13]). Details on Beck-Chevalley conditions are recalled in appendix A, where the general definitions and properties are recalled in section A.1, while their appearance in derived algebraic geometry are recalled in section A.2. Intuitively, this condition means that the square formed by the left adjoints, and the square formed by the right adjoints, are both commutative, and that the vertical functors are compatible with the units and co-units of these adjunctions. In this section, this condition will allow us to prove on the one hand that the adjunction $\mathcal{O}_{\epsilon-gr} \dashv \mathrm{Spec}^{\epsilon-gr}$ does restrict to an adjunction between derived foliations and perfect linear stacks, and on the other hand that checking that the unit and co-unit of this adjunction are equivalences reduces to checking that it is the case for the unit and co-unit of $\mathcal{O}_{gr} \dashv \mathrm{Spec}^{gr}$, which in turn will follow from results of [Mon21, Cor. 2.9, Prop. 2.11]. We will start this section by applying the results of appendix A.

Remark 5.2.1. Consider the maps π and p in the canonical span of conclusion 4.3.8:

$$[\mathcal{L}^{gr}(Z/X)/\mathcal{H}] \xleftarrow{\pi} [\mathcal{L}^{gr}(Z/X)/\mathbb{G}_m] \xrightarrow{p} Z \times B\mathbb{G}_m$$

which form the following two squares, which a priori need not commute:

$$\begin{tikzcd}
\epsilon - cdga^{gr}(Z/X)^{\mathrm{op}} \arrow[r, phantom, "\perp"] \arrow[r, shift right=1.5ex, "\mathrm{Spec}^{\epsilon-gr}"'] \arrow[d, "\pi^*"] & \mathrm{dSt}\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H} \arrow[l, shift right=1.5ex, "\mathcal{O}_{\epsilon-gr}"'] \arrow[d, "\pi^*"] \\
\mathbf{DR}(Z/X)/cdga^{gr}(Z)^{\mathrm{op}} \arrow[r, phantom, "\perp"] \arrow[r, shift right=1.5ex, "\mathrm{Spec}_{\mathbf{DR}-gr}"'] \arrow[d, "p_*"] & \mathrm{dSt}\,/\mathcal{L}^{gr}(Z/X) - \mathbb{G}_m \arrow[l, shift right=1.5ex, "\mathcal{O}_{\mathbf{DR}-gr}"'] \arrow[d, "p_!"] \\
cdga^{gr}(Z)^{\mathrm{op}} \arrow[r, phantom, "\perp"] \arrow[r, shift right=1.5ex, "\mathrm{Spec}^{gr}"'] & \mathrm{dSt}\,/Z - \mathbb{G}_m \arrow[l, shift right=1.5ex, "\mathcal{O}_{gr}"']
\end{tikzcd}$$

In these squares, by the last point of reminder 5.1.4, so far we only know that the following squares commute:

$$\begin{tikzcd}
\epsilon - cdga^{gr}(Z/X)^{\mathrm{op}} \arrow[r, "\mathrm{Spec}_{[\mathcal{L}^{gr}(Z/X)/\mathcal{H}]}"] \arrow[d, "\pi^*"'] & \mathrm{dSt}\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H} \arrow[d, "\pi^*"] \\
cdga^{gr}(\mathbf{DR}(Z/X))^{\mathrm{op}} \arrow[r, "\mathrm{Spec}_{[\mathcal{L}^{gr}(Z/X)/\mathbb{G}_m]}"'] & \mathrm{dSt}\,/\mathcal{L}^{gr}(Z/X) - \mathbb{G}_m
\end{tikzcd}$$

$$\begin{tikzcd}
cdga^{gr}(\mathbf{DR}(Z/X))^{\mathrm{op}} \arrow[d, "p_*"'] & \mathrm{dSt}\,/\mathcal{L}^{gr}(Z/X) - \mathbb{G}_m \arrow[l, "\mathcal{O}_{[\mathcal{L}^{gr}(Z/X)/\mathbb{G}_m]}"'] \arrow[d, "p_!"] \\
cdga^{gr}(Z)^{\mathrm{op}} & \mathrm{dSt}\,/Z - \mathbb{G}_m \arrow[l, "\mathcal{O}_{[Z/\mathbb{G}_m]}"]
\end{tikzcd}$$

We will apply conclusion A.2.12 to obtain that the restriction of these squares to relative affines on the right-hand side are Beck-Chevalley in the sense recalled in reminder A.1.2. As such, we will first prove that p and π satisfy the conditions of this property.

Lemma 5.2.2. *The maps p and π have the following properties:*

1. *p is a quotient of a map between derived stacks relatively affine over* Spec(k) *by an affine group scheme.*
2. *π is a finite map.*

Proof. 1. By definition, p is the quotient of the following map:

$$\mathcal{L}^{gr}(Z/X) = \mathrm{Spec}_Z(\mathbf{DR}(Z/X)) \to Z$$

by the affine group scheme $\mathbb{G}_m$. Moreover, $Z = \mathrm{Spec}(B)$ is affine and thus $\mathrm{Spec}_Z(\mathbf{DR}(Z/X))$ is relatively affine over $\mathrm{Spec}(k)$ too (see remark 4.3.6).

2. Firstly, recall from definition 4.3.2 that $\mathcal{H}$ is defined by the following split short exact sequence of abelian group schemes:

$$1 \to B\mathbb{G}_a \to \mathcal{H} \leftrightarrows \mathbb{G}_m \to 1$$

We will call $\iota : \mathbb{G}_m \to \mathcal{H}$ the section appearing in this diagram, which is a group morphism and thus admits a delooping $B(\iota) : B\mathbb{G}_m \to B\mathcal{H}$. Also note that the projection $\pi : [\mathcal{L}^{gr}(Z/X)/\mathbb{G}_m] \to [\mathcal{L}^{gr}(Z/X)/\mathcal{H}]$ sits in the following pull-back square:

$$\begin{tikzcd}
{[\mathcal{L}^{gr}(Z/X)/\mathbb{G}_m]} \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & B\mathbb{G}_m \arrow[d, "B(\iota)"] \\
{[\mathcal{L}^{gr}(Z/X)/\mathcal{H}]} \arrow[r] & B\mathcal{H}
\end{tikzcd}$$

Therefore, since π is the pull-back of $B(\iota) : B\mathbb{G}_m \to B\mathcal{H}$, it suffices to show that $B(\iota)$ is finite. To see that $B(\iota) : B\mathbb{G}_m \to B\mathcal{H}$ is finite, let us first look at the embedding $\iota : \mathbb{G}_m \to \mathcal{H}$, of which it is the delooping. We will show that ι has fibers equivalent to

$\mathbb{G}_a$. Since it is a group morphism, it suffices to check the fiber over the unit of $\mathcal{H}$, since all fibers are equivalent in this case. Let us thus consider the following diagram:

$$\begin{tikzcd}
\mathbb{G}_a \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "1_{\mathcal{H}}"] & & \\
\mathbb{G}_m \arrow[r, hook, "\iota"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{G}_m \rtimes B\mathbb{G}_a \arrow[d, two heads] & =: & \mathcal{H} \\
\bullet \arrow[r, "1_{B\mathbb{G}_a}"'] & B\mathbb{G}_a & &
\end{tikzcd}$$

Note that in this diagram not all maps are group morphisms, so this diagram is considered inside derived stacks only. Inside this diagram:

– The projection map $\mathcal{H} \to B\mathbb{G}_a$, which is not a group morphism, comes from the identification $\mathcal{H} \simeq \mathbb{G}_m \times B\mathbb{G}_a$ as derived stacks (but the group structure of this derived stack is twisted by the action of $\mathbb{G}_m$, as indicated by the direct product $\rtimes$ in the definition of $\mathcal{H}$).

– The bottom square is a pull-back since the embedding $\iota : \mathbb{G}_m \to \mathcal{H}$ and the projection $\mathcal{H} \to B\mathbb{G}_a$ do form a fiber sequence whose image is exactly the unit element of $B\mathbb{G}_a$.

– The outer square is a pull-back by definition of $B\mathbb{G}_a$.

Therefore, by pasting, the top square is a pull-back, i.e. the fiber of the embedding $\iota : \mathbb{G}_m \to \mathcal{H}$ is equivalent to $\mathbb{G}_a$. Moreover, by commutativity of the top square, the map $\mathbb{G}_a \to \mathbb{G}_m$ is constant equal to the unit of $\mathbb{G}_m$, and is thus a group morphism. As such, we can apply delooping, and in conclusion all the the fibers of the map $B\mathbb{G}_m \to B\mathcal{H}$ are equivalent to $B\mathbb{G}_a$, which is relatively affine and finite over $\mathrm{Spec}(k)$ as it is the relative spectrum of $k[\eta] = k \oplus k[-1]$ in characteristic zero, as noted in [MRT22, §1.1, (A)]. This implies that the map $B\mathbb{G}_m \to B\mathcal{H}$ is finite.

Alternatively, while the above proof only used elementary facts about $\mathcal{H}$, note that using [Rob24, Rmk. 2.3.65], there is a canonical identification $B\mathcal{H} = [B^2\mathbb{G}_a/\mathbb{G}_m]$, and thus we can form the following diagram:

$$\begin{tikzcd}
B\mathbb{G}_a \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B\mathbb{G}_m \arrow[r, equal] \arrow[d] & {[\bullet/\mathbb{G}_m]} \arrow[d] \\
\bullet \arrow[r] & B^2\mathbb{G}_a \arrow[r] & B\mathcal{H} \arrow[r, equal] & {[B^2\mathbb{G}_a/\mathbb{G}_m]}
\end{tikzcd}$$

which directly proves that fibers of $B\mathbb{G}_m \to B\mathcal{H}$ are equivalent to $B\mathbb{G}_a$. ☐

Applying conclusion A.2.12 to the previous lemma, we obtain the following property.

Corollary 5.2.3. *The two inner and thus the outer square in the following diagram are Beck-Chevalley:*

$$\begin{tikzcd}
\epsilon - cdga^{gr}(Z/X)^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}^{\epsilon-gr}"'] \arrow[r, phantom, "\perp"] \arrow[d, "\pi^*"] \arrow[dd, bend right=50, "-^{gr}"'] & \mathrm{RelAff}/\mathcal{L}^{gr}(Z/X) - \mathcal{H} \arrow[l, shift right, "\mathcal{O}_{\epsilon-gr}"'] \arrow[d, "\pi^*"] \arrow[dd, bend left=50, "-^{Z-\mathbb{G}_m}"] \\
\mathbf{DR}(Z/X)/cdga^{gr}(Z)^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_{\mathbf{DR}-gr}"'] \arrow[r, phantom, "\perp"] \arrow[d, "p_*"] & \mathrm{RelAff}/\mathcal{L}^{gr}(Z/X) - \mathbb{G}_m \arrow[l, shift right, "\mathcal{O}_{\mathbf{DR}-gr}"'] \arrow[d, "p_!"] \\
cdga^{gr}(Z)^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}^{gr}"'] \arrow[r, phantom, "\perp"] & \mathrm{RelAff}/Z - \mathbb{G}_m \arrow[l, shift right, "\mathcal{O}_{gr}"']
\end{tikzcd}$$

In particular, the two following diagrams commute:

1.

$$\begin{tikzcd}
\epsilon - cdga^{gr}(Z/X)^{\mathrm{op}} \arrow[r, "\mathrm{Spec}^{\epsilon-gr}"'] \arrow[d, "-^{gr}"'] & \mathrm{RelAff}/\mathcal{L}^{gr}(Z/X) - \mathcal{H} \arrow[d, "-^{Z-\mathbb{G}_m}"] \\
cdga^{gr}(Z)^{\mathrm{op}} \arrow[r, "\mathrm{Spec}^{gr}"'] & \mathrm{RelAff}/Z - \mathbb{G}_m
\end{tikzcd}$$

2.

$$\begin{tikzcd}
\epsilon - cdga^{gr}(Z/X)^{\mathrm{op}} \arrow[d, "-^{gr}"'] & \mathrm{RelAff}/\mathcal{L}^{gr}(Z/X) - \mathcal{H} \arrow[l, "\mathcal{O}_{\epsilon-gr}"'] \arrow[d, "-^{Z-\mathbb{G}_m}"] \\
cdga^{gr}(Z)^{\mathrm{op}} & \mathrm{RelAff}/Z - \mathbb{G}_m \arrow[l, "\mathcal{O}_{gr}"']
\end{tikzcd}$$

Using this corollary, we will first show that the adjunction $\mathcal{O}_{\epsilon-gr} \dashv \mathrm{Spec}^{\epsilon-gr}$ restricts to $\mathcal{F}ol(Z/X)$ on the left-hand side and $\mathrm{PerfLinSt}_Z\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H}$ on the right-hand side.

Lemma 5.2.4. *Let* $F \in \epsilon - cdga^{gr}(Z/X)$*, and* $V := \mathrm{Spec}^{\epsilon-gr}(F) \in \mathrm{RelAff}/\mathcal{L}^{gr}(Z/X)$ $-\mathcal{H}$.
If $F \in \mathcal{F}ol(Z/X)$*, then* $V \in \mathrm{PerfLinSt}_Z\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H}$.

Proof. By definition, it suffices to prove that V is equivalent to $\mathbb{V}(L)$ for some $L \in \mathrm{Perf}(Z)$ when viewed as an object in $\mathrm{dSt}\,/Z - \mathbb{G}_m$ after applying $-^{Z-\mathbb{G}_m}$. By the commutativity of square 1. in corollary 5.2.3, we have that viewing V in this ∞-category is equivalent to computing $\mathrm{Spec}^{gr}(F^{gr})$, where F^{gr} is the underlying graded algebra of F. Since F is a derived foliation by assumption, we have that $F^{gr} \simeq \mathrm{Sym}_{\mathcal{O}_Z} L$ as graded algebras for some $L \in \mathrm{Perf}(Z)$, so that $V \simeq \mathbb{V}(L)$ as $\mathbb{G}_m$-equivariant derived stacks over Z as required. ☐

Lemma 5.2.5. *Let* $V \in \mathrm{dSt}\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H}$*, and let* $F := \mathcal{O}_{\epsilon-gr}V \in \epsilon - cdga^{gr}(Z/X)$.
If $V \in \mathrm{PerfLinSt}_Z\,/\mathcal{L}^{gr}(Z/X) - \mathcal{H}$*, then* $F \in \mathcal{F}ol(Z/X)$.

Proof. Similarly, in this case $V^{Z-\mathbb{G}_m} \simeq \mathbb{V}(L)$ for some $L \in \mathrm{Perf}(Z)$ by assumption, and by the commutativity of square 2. in corollary 5.2.3, we have that the graded algebra underlying F is equivalent as a graded algebra to $\mathcal{O}_{gr}\mathbb{V}(L)$. Since L is perfect, by [Mon21, Prop. 2.11], we obtain that $\mathcal{O}_{gr}\mathbb{V}(L) \simeq \mathrm{Sym}_{\mathcal{O}_Z} L$ as graded algebras. This implies that F is quasi-free, and that its weight-1 part $F(1)$ as in definition 4.2.17, which is thus L here, is perfect. Therefore, F is a derived foliation as required. ☐

Corollary 5.2.6. *The adjoint functors* $\mathcal{O}_{\epsilon-gr} \dashv \operatorname{Spec}^{\epsilon-gr}$ *restrict to an adjunction:*

$$\mathcal{F}ol(Z/X) \underset{\operatorname{Spec}^{\epsilon-gr}}{\overset{\mathcal{O}_{\epsilon-gr}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \operatorname{PerfLinSt}_Z /\mathcal{L}^{gr}(Z/X) - \mathcal{H}$$

We will now show that it is an equivalence of ∞-categories.

Lemma 5.2.7. *The co-unit of* $\mathcal{O}_{\epsilon-gr} \dashv \operatorname{Spec}^{\epsilon-gr}$ *is an equivalence on objects that belong to* $\mathcal{F}ol(Z/X)$.

Proof. Let $F \in \mathcal{F}ol(Z/X)$. By lemma A.1.5 applied to the outer square of corollary 5.2.3, the co-unit of the adjunction $\mathcal{O}_{\epsilon-gr} \operatorname{Spec}^{\epsilon-gr}(F) \leftarrow F$ is mapped by the forgetful functor $-^{gr} : \epsilon - cdga^{gr}(Z/X) \to cdga^{gr}(Z)$ to the co-unit of the adjunction $\mathcal{O}_{gr} \operatorname{Spec}^{gr}(F) \leftarrow F$. By [Mon21, Cor. 2.9], this map is an equivalence whenever F is quasi-free and $F(1) \in \operatorname{QCoh}^-(Z)$, i.e. $F(1)$ is left-bounded in amplitude. Since here F is a derived foliation, $F(1)$ is perfect and thus left-bounded, so this co-unit is an equivalence. Finally, since $-^{gr}$ is a conservative functor, the initial co-unit is an equivalence too. ☐

Lemma 5.2.8. *The unit of* $\mathcal{O}_{\epsilon-gr} \dashv \operatorname{Spec}^{\epsilon-gr}$ *is an equivalence on objects that belong to* $\operatorname{PerfLinSt}_Z /\mathcal{L}^{gr}(Z/X) - \mathcal{H}$.

Proof. Similarly, for $V = \mathbb{V}(L) \in \operatorname{PerfLinSt}_Z /\mathcal{L}^{gr}(Z/X) - \mathcal{H}$, the unit $V \to \operatorname{Spec}^{\epsilon-gr}(\mathcal{O}_{\epsilon-gr}V)$ is sent by the forgetful functor $\operatorname{PerfLinSt}_Z /\mathcal{L}^{gr}(Z/X) - \mathcal{H} \to \operatorname{dSt}/Z - \mathbb{G}_m$ to the unit $V \to \operatorname{Spec}^{gr}(\mathcal{O}_{gr}V)$. By [Mon21, Prop. 2.11], since L is perfect, this last map is an equivalence, but the forgetful functor is conservative, so the former unit also is an equivalence. ☐

Theorem 5.2.9. *The following adjunction is an equivalence of* ∞*-categories:*

$$\mathcal{F}ol(Z/X) \underset{\operatorname{Spec}^{\epsilon-gr}}{\overset{\mathcal{O}_{\epsilon-gr}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \operatorname{PerfLinSt}_Z /\mathcal{L}^{gr}(Z/X) - \mathcal{H}$$

5.3 Equivalence in the Global Case

To finish this chapter, after dealing with the affine case, we will now globalize these results by inspecting how theorem 5.2.9 behaves under gluing. This will yield a slightly different equivalence in the general case, but we will show that it reduces to the same equivalence when $Z \to X$ is a relative derived Deligne-Mumford stack.

5.3.1 Structures in the Global Case

Let us start by defining structures from the previous section in the global case. This will require first inspecting the functoriality of these categories on affines. Note that by remark 4.2.19, the ∞-category $\mathcal{F}ol$ of perfect foliations is not functorial in general, but it is functorial when restricted to affines $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ which are of finite presentation. This motivates the following remark.

Remark 5.3.1. Let $p : Z \to X$ be an arbitrary map of derived stacks. We denote by $\mathrm{Arr}(\mathrm{dAff})/p$ the ∞-category of commutative squares:

$$\begin{tikzcd} V \arrow[r] \arrow[d] & Z \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

where U and V are derived affines. This is the canonical generating site of the ∞-topos $\mathrm{Arr}(\mathrm{dSt})$ as in example 3.2.57, and as such the colimit of the objects $V \to U$ in this category is always equivalent to p.

We now define $\mathrm{Arr}(\mathrm{dAff})^{fp}/p$ to be the full subcategory of $\mathrm{Arr}(\mathrm{dAff})/p$ spanned by the objects as above such that the map $V \to U$ is of finite presentation. In general, it is no longer the case that colimits of such objects recover p, but we will show that it is the case when p is locally of finite presentation.

Proposition 5.3.2. *Let $p : Z \to X$ be a map of derived stacks which is locally of finite presentation in the sense of [LurieDAG14, Def. 2.3.1]. Then $\mathrm{Arr}(\mathrm{dAff})^{fp}/p$ is cofinal in $\mathrm{Arr}(\mathrm{dAff})/p$, and thus $f \simeq \operatorname*{colim}_{(V \to U) \in \mathrm{Arr}(\mathrm{dAff})^{fp}/p} (V \to U)$.*

Proof. Set

$$\mathcal{C} := \mathrm{Arr}(\mathrm{dAff})/p, \qquad \mathcal{C}^{fp} := \mathrm{Arr}(\mathrm{dAff})^{fp}/p.$$

By Quillen's Theorem A in the ∞-categorical form [HTT, Th. 4.1.3.1], it suffices to show that for every object

$$\xi = \Big(V \to Z,\ U \to X,\ V \to U\Big) \in \mathcal{C},$$

the slice $(\mathcal{C}^{fp})_{\xi/}$ is weakly contractible.

Let

$$Z_U := Z \times_X U,$$

and let $g\colon V \to Z_U$ be the induced U-morphism. Since p is locally of finite presentation, so is

$$p_U\colon Z_U \to U$$

by base change stability [LurieDAG14, Prop. 2.3.4].

Let $\mathcal{D}_\xi \subset (\mathcal{C}^{fp})_{\xi/}$ be the full subcategory consisting of those objects whose base is U. Concretely, an object of $\mathcal{D}_\xi$ is a factorization

$$V \xrightarrow{\alpha} W \xrightarrow{\beta} Z_U$$

with W derived affine and $W \to U$ of finite presentation.

The inclusion

$$i\colon \mathcal{D}_\xi \hookrightarrow (\mathcal{C}^{fp})_{\xi/}$$

admits a right adjoint

$$r\colon (\mathcal{C}^{fp})_{\xi/} \to \mathcal{D}_\xi$$

given by pullback along $U \to U'$: if

$$\eta = \Big(\xi \to (W \to U') \to p\Big),$$

then

$$r(\eta) := \Big(V \to W \times_{U'} U \to Z_U\Big).$$

The adjunction is the universal property of the fiber product. Hence i induces an equivalence on classifying spaces, so it is enough to show that $\mathcal{D}_\xi$ is weakly contractible.

Now view Z_U as a prestack on $\mathrm{dAff}_{/U}$. Since p_U is locally of finite presentation, the functor

$$F_U\colon (\mathrm{dAff}_{/U})^{op} \to \mathcal{S}, \qquad T \mapsto \mathbf{Map}_U(T, Z_U)$$

commutes with filtered colimits of connective U-algebras; equivalently, F_U is the left Kan extension of its restriction to affine U-schemes of finite presentation, by [LurieDAG14, Rmk. 2.3.5(a)]. Therefore, for the affine $V \to U$,

$$\mathbf{Map}_U(V, Z_U) \simeq \operatorname*{colim}_{(V\to W)\in(\mathrm{dAff}^{fp}_{/U})^{\mathrm{op}}_{V/}} \mathbf{Map}_U(W, Z_U).$$

Since the category $\mathcal{D}_\xi$ is the ∞-category of pairs

$$\Big((V \to W),\ \tilde{g}\colon W \to Z_U\Big)$$

whose restriction to V is g, it identifies with the fiber over g of the canonical map

$$\int_I F \to \operatorname*{colim}_I F \to \mathbf{Map}_U(V, Z_U),$$

where $I := \Big(\mathrm{dAff}^{fp}_{/U}\Big)^{op}_{V/}$, $F(V \to W) := \mathbf{Map}_U(W, Z_U)$, and $\int_I F$ denotes the Grothendieck construction. Since the classifying space of $\int_I F$ is $\mathrm{colim}_I\, F$, which is equivalent to $\mathbf{Map}_U(V, Z_U)$ by the above, this fiber is contractible. Hence $\mathcal{D}_\xi$ is weakly contractible, and so is $(\mathcal{C}^{fp})_{\xi/}$.

Applying [HTT, Thm. 4.1.3.1], the inclusion j is cofinal. ☐

Remark 5.3.3. In fact, we expect the above proposition to be an equivalence, i.e. a map p is locally of finite presentation if and only if $\mathrm{Arr}(\mathrm{dAff})^{fp}/p$ is cofinal in $\mathrm{Arr}(\mathrm{dAff})/p$, but we will not need this fact here.

Definition 5.3.4. *Let $p : Z \to X$ be an arbitrary map of derived stacks.*

1. *By construction 4.2.14, we defined contravariant functors* $\epsilon - dg^{gr}, \epsilon - cdga^{gr}$: $\mathrm{Arr}(\mathrm{dAff})^{\mathrm{op}} \to \infty - Cat$. *By remark 4.2.19, we can restrict* $\epsilon - cdga^{gr}$ *to a contravariant functor* $\mathcal{F}ol$: $\mathrm{Arr}(\mathrm{dAff})^{fp,\mathrm{op}} \to \infty - Cat$ *on the finitely-presented objects. Similarly, by pull-back of graded algebras, we get contravariant functors* $dg^{gr}, cdga^{gr}$: $\mathrm{Arr}(\mathrm{dAff})^{\mathrm{op}} \to \infty - Cat$ *which map* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *to* $dg^{gr}(B), cdga^{gr}(B)$.
2. *Analogously to [TV25a, Def. 2.1.3], when* $p : Z \to X$ *is moreover locally of finite presentation*[1]*, we set:*

$$\mathcal{F}ol(Z/X) := \lim_{(V\to U)\in\left(\mathrm{Arr}(\mathrm{dAff})^{fp}/p\right)^{\mathrm{op}}} \mathcal{F}ol(V/U)$$

3. *Similarly, we set:*

$$\epsilon - dg^{gr}(Z/X) := \lim_{(V\to U)\in(Arr(\mathrm{dAff})/p)^{\mathrm{op}}} \epsilon - dg^{gr}(V/U)$$
$$\epsilon - cdga^{gr}(Z/X) := \lim_{(V\to U)\in(Arr(\mathrm{dAff})/p)^{\mathrm{op}}} \epsilon - cdga^{gr}(V/U)$$

4. *Finally, we set:*

$$dg^{gr}(Z) := \lim_{V\in(\mathrm{dAff}\,/Z)^{\mathrm{op}}} dg^{gr}(V)$$
$$cdga^{gr}(Z) := \lim_{V\in(\mathrm{dAff}\,/Z)^{\mathrm{op}}} cdga^{gr}(V)$$

Remark 5.3.5. Note that by proposition 5.3.2, in the above definitions, when $p : Z \to X$ is locally of finite presentation, all instances of $Arr(\mathrm{dAff})/p$ can be replaced by $Arr(\mathrm{dAff})^{fp}/p$, as they compute the same limits.

Also note that if Z and X are affines, the above definitions do recover the definitions in the affine case, because in this case the above limits admit an initial object given by taking $(V \to U) = (Z \to X)$.

Moreover, it is important to note that while objects of $dg^{gr}, cdga^{gr}$ are sheaves of graded objects in the usual sense, it is not the case for $\epsilon - dg^{gr}$ and $\epsilon - cdga^{gr}$ because the functoriality of the limit defining them is not the usual one (the pull-back is given by construction 4.2.14, which is more complicated than usual pull-backs of sheaves). This means that at this stage defining an object of $\mathcal{F}ol(Z/X)$ is quite difficult. We will later on establish more simple descriptions of these categories.

Remark 5.3.6. In this remark, let $p : Z \to X$ be an arbitrary map of derived stacks, not necessarily locally of finite presentation. Consider the definition of the above point 3. For $(V \to U) \in \mathrm{Arr}(\mathrm{dAff})/p$, by remark 4.3.6 we have:

$$\epsilon - dg^{gr}(V/U) \simeq \mathrm{QCoh}([\mathcal{L}^{gr}(V/U)/\mathcal{H}])$$

---

[1] We expect the definition of $\mathcal{F}ol$ to extend beyond the locally finitely presented case.

As a consequence, since Arr(dAff)$/p$ covers $Z \to X$ by example 3.2.57, and QCoh sends colimits to limits by proposition 3.4.11, we obtain that:

$$\epsilon - dg^{gr}(Z/X) \simeq \mathrm{QCoh}(\underset{(V\to U)\in \mathrm{Arr}(\mathrm{dAff})/p}{\mathrm{colim}}[\mathcal{L}^{gr}(V/U)/\mathcal{H}])$$

Moreover, the derived stack $\underset{(V\to U)\in \mathrm{Arr}(\mathrm{dAff})/p}{\mathrm{colim}} \mathcal{L}^{gr}(V/U)$ inherits an $\mathcal{H}$-action, and taking the quotient by $\mathcal{H}$ commutes with colimits (as the quotient by $\mathcal{H}$ is itself a colimit), so we obtain that:

$$\epsilon - dg^{gr}(Z/X) \simeq \mathrm{QCoh}([\underset{(V\to U)\in \mathrm{Arr}(\mathrm{dAff})/p}{\mathrm{colim}} \mathcal{L}^{gr}(V/U)/\mathcal{H}])$$

This motivates the next definition.

Also note that a similar computation in the graded case directly yields the known equivalence:

$$dg^{gr}(Z) \simeq \mathrm{QCoh}(Z \times B\mathbb{G}_m)$$

Definition 5.3.7. *Let* $\widetilde{\mathcal{L}^{gr}}(Z/X) := \underset{(V\to U)\in \mathrm{Arr}(\mathrm{dAff})/p}{\mathrm{colim}} \mathcal{L}^{gr}(V/U)$*. It comes equipped with a canonical* $\mathcal{H}$*-action, and a canonical* $\mathcal{H}$*-equivariant map* $\widetilde{\mathcal{L}^{gr}}(Z/X) \to \mathcal{L}^{gr}(Z/X)$ *over* Z*, which we do not expect to be an equivalence in general.*

Remark 5.3.8. In [BN12, Def. 1.22], another object called $\widehat{\mathcal{L}}$ is defined as the formal completion of the loop stack at the constant loops. Note that this is a distinct object, although we expect them to coincide at least in some cases, but we will not need this fact here.

Corollary 5.3.9. *Let* $Z \to X$ *be an arbitrary map of derived stacks. By remark 5.3.6, we have the following equivalences:*

$$\begin{aligned}
\epsilon - dg^{gr}(Z/X) &\simeq \mathrm{QCoh}([\widetilde{\mathcal{L}^{gr}}(Z/X)/\mathcal{H}]) \\
\epsilon - cdga^{gr}(Z/X) &\simeq \mathrm{CAlg}(\mathrm{QCoh}([\widetilde{\mathcal{L}^{gr}}(Z/X)/\mathcal{H}])) \\
dg^{gr}(Z) &\simeq \mathrm{QCoh}(Z \times B\mathbb{G}_m) \\
cdga^{gr}(Z) &\simeq \mathrm{CAlg}(\mathrm{QCoh}(Z \times B\mathbb{G}_m))
\end{aligned}$$

Remark 5.3.10. Note that it is immediate to see that the functor $(V \to U) \mapsto V \times B\mathbb{G}_m$: Arr(dAff) $\to$ dSt satisfies étale descent, and that its left Kan extension is simply given by $(Z \to X) \mapsto Z \times B\mathbb{G}_m$. Since QCoh and CAlg preserve étale descent, we also obtain that $dg^{gr}, cdga^{gr} : \mathrm{Arr}(\mathrm{dAff})^{\mathrm{op}} \to \infty - Cat$ are stacks for the étale topology. In particular, we obtain that dg^{gr} and $cdga^{gr}$ from definition 5.3.4 send arbitrary colimits to limits.

Similarly, to obtain that $\epsilon - dg^{gr}, \epsilon - cdga^{gr} : \mathrm{Arr}(\mathrm{dAff})^{\mathrm{op}} \to \infty - Cat$ are stacks for the étale topology, and that they send arbitrary colimits to limits, it suffices to prove that the functor $(V \to U) \mapsto \mathcal{L}^{gr}(V/U)$: Arr(dAff) $\to$ dSt satisfies étale codescent. This is the content of the following proposition.

Proposition 5.3.11. *The functor* $(V \to U) \mapsto \mathcal{L}^{gr}(V/U) : \mathrm{Arr}(\mathrm{dAff}) \to \mathrm{dSt}$ *satisfies étale codescent.*

Proof. Let

$$u : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

be an object of Arr(dAff), and let

$$u_\bullet : \mathrm{Spec}(B_\bullet) \to \mathrm{Spec}(A_\bullet)$$

be an étale hypercover of u in Arr(dAff). Thus $u_\bullet$ is an augmented simplicial object over u, and the augmentations

$$\mathrm{Spec}(B_\bullet) \to \mathrm{Spec}(B), \qquad \mathrm{Spec}(A_\bullet) \to \mathrm{Spec}(A)$$

are étale hypercovers.

In particular, for every $n \geq 0$, the morphisms

$$\mathrm{Spec}(B_n) \to \mathrm{Spec}(B), \qquad \mathrm{Spec}(A_n) \to \mathrm{Spec}(A)$$

are étale, $\mathrm{Spec}(B) \simeq \mathrm{colim}\ \mathrm{Spec}(B_\bullet)$ and $\mathrm{Spec}(A) \simeq \mathrm{colim}\ \mathrm{Spec}(A_\bullet)$. Let $n \in \mathbb{N}$. Since the maps $A \to A_n$ and $B \to B_n$ are étale, we have:

$$\mathbb{L}_{B_n/B} \simeq 0, \qquad \mathbb{L}_{A_n/A} \simeq 0.$$

Applying transitivity to the composable maps $A \to B \to B_n$, one gets a triangle

$$B_n \otimes_B \mathbb{L}_{B/A} \to \mathbb{L}_{B_n/A} \to \mathbb{L}_{B_n/B}$$

Since $\mathbb{L}_{B_n/B} \simeq 0$, this implies that $B_n \otimes_B \mathbb{L}_{B/A} \xrightarrow{\sim} \mathbb{L}_{B_n/A}$. Similarly, applying transitivity again to $A \to A_n \to B_n$, one gets

$$\mathbb{L}_{B_n/A} \xrightarrow{\sim} \mathbb{L}_{B_n/A_n}.$$

Therefore

$$\mathbb{L}_{B_n/A_n} \simeq B_n \otimes_B \mathbb{L}_{B/A}.$$

It follows that

$$\mathrm{Sym}_{B_n}\Big(\mathbb{L}_{B_n/A_n}[1]\Big) \simeq \mathrm{Sym}_{B_n}\Big((B_n \otimes_B \mathbb{L}_{B/A})[1]\Big) \simeq B_n \otimes_B \mathrm{Sym}_B\Big(\mathbb{L}_{B/A}[1]\Big).$$

By compatibility of relative spectrum with base change, this yields

$$\mathcal{L}^{gr}(u_n) \simeq \mathcal{L}^{gr}(u) \times_{\mathrm{Spec}(B)} \mathrm{Spec}(B_n)$$

functorially in n. Passing to the colimit, we obtain:

$$\operatorname{colim} \mathcal{L}^{gr}(u_\bullet) \simeq \operatorname{colim} \mathcal{L}^{gr}(u) \times_{\operatorname{Spec}(B)} \operatorname{Spec}(B_\bullet)$$

Since colimits commute with pull-backs, we obtain:

$$\operatorname{colim} \mathcal{L}^{gr}(u_\bullet) \simeq \mathcal{L}^{gr}(u) \times_{\operatorname{Spec}(B)} \operatorname{Spec}(B) \simeq \mathcal{L}^{gr}(u)$$

Thus $\mathcal{L}^{gr}$ satisfies étale codescent. ☐

Remark 5.3.12. Note that $\widetilde{\mathcal{L}^{gr}} : \operatorname{Arr}(\mathrm{dSt}) \to \mathrm{dSt}$ is by definition the left Kan extension of $\mathcal{L}^{gr} : \operatorname{Arr}(\mathrm{dAff}) \to \mathrm{dSt}$. As such, for affines $V \to U$, the canonical map $\widetilde{\mathcal{L}^{gr}}(V/U) \to \mathcal{L}^{gr}(V/U)$ is an equivalence. Over a fixed affine base U, this also extends immediately to a disjoint union of affines because mapping stacks commute with disjoint union in their target, therefore: $\mathcal{L}^{gr}(\coprod_\bullet V_\bullet/U) = \underline{\mathbf{Map}}_U(B\mathbb{G}_a \times U, \coprod_\bullet V_\bullet) \simeq \coprod_\bullet \underline{\mathbf{Map}}_U(B\mathbb{G}_a \times U, V_\bullet) \simeq \coprod_\bullet \mathcal{L}^{gr}(V_\bullet/U)$. Also, since by construction this is a map of derived stacks over V, and it is $\mathcal{H}$-equivariant, these derived stacks are equivalent as derived stacks over V and as $\mathcal{H}$-equivariant derived stacks over U.

Additionally, as a left Kan extension of a functor satisfying étale codescent on affines, it commutes with all colimits in $\operatorname{Arr}(\mathrm{dSt})$. Equivalently, for any diagram $(Z_i \to X_i)$ in $\operatorname{Arr}(\mathrm{dSt})$ with colimit $Z \to X$, one has $\widetilde{\mathcal{L}^{gr}}(Z/X) \simeq \operatorname{colim}_i \widetilde{\mathcal{L}^{gr}}(Z_i/X_i)$.

Remark 5.3.13. Also note, as in remark 5.3.10, that the above proposition also implies that $\epsilon - dg^{gr}, \epsilon - cdga^{gr} : \operatorname{Arr}(\mathrm{dAff})^{\mathrm{op}} \to \infty - Cat$ are stacks for the étale topology.

Moreover, when $p : Z \to X$ is a map of derived stacks locally of finite presentation, the fact that $\operatorname{Arr}(\mathrm{dAff})^{fp}/p$ is cofinal in $\operatorname{Arr}(\mathrm{dAff})/p$ by proposition 5.3.2 implies that $\epsilon - cdga^{gr}(Z/X) \simeq \lim_{(V\to U)\in(\operatorname{Arr}(\mathrm{dAff})^{fp}/p)^{\mathrm{op}}} \epsilon - cdga^{gr}(V/U)$, and the fact that $\mathcal{F}ol$ is stable by the functoriality of this diagram by remark 4.2.19 also implies that $\mathcal{F}ol : (\operatorname{Arr}(\mathrm{dAff})^{fp}/p)^{\mathrm{op}} \to \infty - Cat$ satisfies descent for the étale topology restricted to this ∞-category.

5.3.2 Equivalence for Maps Locally of Finite Presentation

To generalize the equivalence of the affine case, we will start by proving that it is stable under affine base-change.

Proposition 5.3.14. *Let $Z \to X \in \operatorname{Arr}(\mathrm{dAff})^{fp}$. The equivalence of ∞-categories $\mathcal{F}ol(Z/X) \xrightarrow[\sim]{\operatorname{Spec}^{\epsilon-gr}} \operatorname{PerfLinSt}_Z /\mathcal{L}^{gr}(Z/X) - \mathcal{H}$ provided by theorem 5.2.9 is contravariant in $Z \to X$, i.e. it defines a functor $\operatorname{Arr}(\mathrm{dAff})^{fp,\mathrm{op}} \to \operatorname{Equiv}(\infty - Cat)$.*

Proof. We have already described the functoriality of $\mathcal{F}ol(Z/X)$ in definition 5.3.4. We will now describe the functoriality of $(Z \to X) \mapsto \mathrm{PerfLinSt}_Z / \mathcal{L}^{gr}(Z/X) - \mathcal{H}$. Consider a commutative square where each object is affine, and both $Z \to X$ and $Z' \to X'$ are of finite presentation:

$$\begin{tikzcd}
Z' \arrow[r] \arrow[d] & Z \arrow[d] \\
X' \arrow[r] & X
\end{tikzcd}$$

By functoriality of $\mathcal{L}^{gr}$, it induces a canonical map $p : \mathcal{L}^{gr}(Z'/X') \to \mathcal{L}^{gr}(Z/X)$ of derived stacks over Z. We will describe the pull-back functor p^* on the side of derived stacks. Let $\mathbb{V}(E[1]) \in \mathrm{PerfLinSt}_Z / \mathcal{L}^{gr}(Z/X) - \mathcal{H}$. Since all the objects are affines, we can form the following pull-back diagram:

$$\begin{tikzcd}
\mathrm{Spec}(\mathbf{DR}(Z'/X') \otimes_{\mathbf{DR}(Z/X)} \mathrm{Sym}_{\mathcal{O}_Z} E[1]) \arrow[rr] \arrow[d] \arrow[drr, phantom, very near start, "\lrcorner"] & & \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_Z} E[1]) \arrow[d] \\
\mathcal{L}^{gr}(Z'/X') = \mathrm{Spec}(\mathbf{DR}(Z'/X')) \arrow[rr, "p"'] \arrow[dr] & & \mathrm{Spec}(\mathbf{DR}(Z/X)) = \mathcal{L}^{gr}(Z/X) \arrow[dl] \\
& Z &
\end{tikzcd}$$

Let $Z = \mathrm{Spec}(B)$, $Z' = \mathrm{Spec}(B')$, $X = \mathrm{Spec}(A)$ and $X' = \mathrm{Spec}(A')$, and note that $\mathbf{DR}(Z/X) = \mathrm{Sym}_B(\mathbb{L}_{B/A}[1])$ and $\mathbf{DR}(Z'/X') = \mathrm{Sym}_{B'}(\mathbb{L}_{B'/A'}[1])$. As such, the homotopy fiber product in the top left corner can be computed by derived tensor products, which yields $\mathrm{Sym}_{B'}\left(\left(\mathbb{L}_{B'/A'} \oplus_{\mathbb{L}_{B/A} \otimes_B B'} (E \otimes_B B')\right)[1]\right)$. In particular, since $\mathrm{Spec}(B') \to \mathrm{Spec}(A')$ and $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are of finite presentation by assumption, then $\mathbb{L}_{B'/A'}$ and $\mathbb{L}_{B/A}$ are perfect. Moreover, E is perfect by assumption, and $- \otimes_B B'$ preserves perfects. Therefore, we do obtain that $p^*\mathbb{V}(E[1])$ is a perfect linear stack over Z', equipped with a structural map to $\mathcal{L}^{gr}(Z'/X')$, and p^* does induce an $\mathcal{H}$-action on it, so $p^*\mathbb{V}(E[1]) \in \mathrm{PerfLinSt}_{Z'} / \mathcal{L}^{gr}(Z'/X') - \mathcal{H}$. Finally, by reminder 5.1.4, we obtain that the following square is canonically commutative:

$$\begin{tikzcd}
\mathcal{F}ol(Z/X) \arrow[r, "\simeq", "\mathrm{Spec}^{\epsilon-gr}_{Z/X}"'] \arrow[d, "p^*"'] & \mathrm{PerfLinSt}_Z / \mathcal{L}^{gr}(Z/X) - \mathcal{H} \arrow[d, "p^*"] \\
\mathcal{F}ol(Z'/X') \arrow[r, "\simeq", "\mathrm{Spec}^{\epsilon-gr}_{Z'/X'}"'] & \mathrm{PerfLinSt}_{Z'} / \mathcal{L}^{gr}(Z'/X') - \mathcal{H}
\end{tikzcd}$$

Which precisely means that the map:

$$(Z \to X) \mapsto (\mathcal{F}ol(Z/X) \simeq \mathrm{PerfLinSt}_Z / \mathcal{L}^{gr}(Z/X) - \mathcal{H})$$

is contravariant in $Z \to X$. □

Remark 5.3.15. Consider now $p : Z \to X$ a map of derived stacks which is locally of finite presentation. Starting from the equivalence of theorem 5.2.9, and passing through the limit over $\mathrm{Arr}(\mathrm{dAff})^{fp}/p$ using the functoriality described in the above corollary, we always obtain an equivalence of ∞-categories:

$$\lim_{(V\to U)\in(\mathrm{Arr}(\mathrm{dAff})^{fp}/p)^{\mathrm{op}}} \mathcal{F}ol(V/U) \simeq \lim_{(V\to U)\in(\mathrm{Arr}(\mathrm{dAff})^{fp}/p)^{\mathrm{op}}} \mathrm{PerfLinSt}_V\,/\mathcal{L}^{gr}(V/U)-\mathcal{H}$$

Moreover, on the left-hand side, by definition 2. in definition 5.3.4, it is always the case that:

$$\mathcal{F}ol(Z/X) := \lim_{(V\to U)\in(\mathrm{Arr}(\mathrm{dAff})^{fp}/p)^{\mathrm{op}}} \mathcal{F}ol(V/U)$$

However, on the right-hand side, it is not always the case that:

$$\mathrm{PerfLinSt}_Z\,/\mathcal{L}^{gr}(Z/X)-\mathcal{H} \overset{?}{=} \lim_{(V\to U)\in(\mathrm{Arr}(\mathrm{dAff})^{fp}/p)^{\mathrm{op}}} \mathrm{PerfLinSt}_V\,/\mathcal{L}^{gr}(V/U)-\mathcal{H}$$

Instead, we will describe the right-hand limit in the next proposition.

Proposition 5.3.16. *Let $p : Z \to X$ be a map of derived stacks which is locally of finite presentation. There is a canonical equivalence of ∞-categories:*

$$\lim_{(V\to U)\in(\mathrm{Arr}(\mathrm{dAff})^{fp}/p)^{\mathrm{op}}} \mathrm{PerfLinSt}_V\,/\mathcal{L}^{gr}(V/U)-\mathcal{H} = \mathrm{PerfLinSt}_Z\,/\widetilde{\mathcal{L}^{gr}}(Z/X)-\mathcal{H}$$

Proof. By definition of $\widetilde{\mathcal{L}^{gr}}(Z/X)$ and its $\mathcal{H}$-action, since $\mathrm{Arr}(\mathrm{dAff})^{fp}/p$ is cofinal in $\mathrm{Arr}(\mathrm{dAff})/p$ by proposition 5.3.2, and since $T \mapsto \mathrm{dSt}\,/T$ sends colimits to limits, we have:

$$\lim_{(V\to U)\in(\mathrm{Arr}(\mathrm{dAff})^{fp}/p)^{\mathrm{op}}} \mathrm{dSt}\,/\mathcal{L}^{gr}(V/U)-\mathcal{H} \simeq \mathrm{dSt}\,/\widetilde{\mathcal{L}^{gr}}(Z/X)-\mathcal{H}$$

So it remains to show that:

$$\lim_{(V\to U)\in(\mathrm{Arr}(\mathrm{dAff})^{fp}/p)^{\mathrm{op}}} \mathrm{PerfLinSt}_V \simeq \mathrm{PerfLinSt}_Z$$

Since the functor $\mathrm{Arr}(\mathrm{dAff})/p \to \mathrm{dAff}\,/Z$ which maps $(V \to U)$ to V is cofinal (as it is right adjoint to the functor $(V \to Z) \mapsto (V \xrightarrow{id_V} V) \in \mathrm{Arr}(\mathrm{dAff})/p$, and by applying [Ker26, Tag 02P3]), and since the inclusion $\mathrm{Arr}(\mathrm{dAff})^{fp}/p \subset \mathrm{Arr}(\mathrm{dAff})/p$ is cofinal by proposition 5.3.2, then by composition the functor $\mathrm{Arr}(\mathrm{dAff})^{fp}/p \to \mathrm{dAff}\,/Z$ is cofinal too. Therefore, it suffices to prove that:

$$\lim_{V\in(\mathrm{dAff}\,/Z)^{\mathrm{op}}} \mathrm{PerfLinSt}_V \simeq \mathrm{PerfLinSt}_Z$$

Note that in this case, for $v : V' \to V \in \mathrm{dAff}\,/Z$, the functoriality $\mathrm{PerfLinSt}_V \to \mathrm{PerfLinSt}_{V'}$ is given $\mathbb{V}_V(E) \mapsto v^*\mathbb{V}_V(E) \simeq \mathbb{V}_{V'}(v^*E)$ by reminder 5.1.4 and since v^* commutes with Sym because v^* is monoidal.

Since Perf satisfies descent by [HAGII, Cor. 1.3.7.4] (see proposition 3.4.13), we have:

$$\mathrm{Perf}(Z) \simeq \lim_{V \in (\mathrm{dAff}\,/Z)^{\mathrm{op}}} \mathrm{Perf}(V)$$

Moreover, by [Mon21, Cor. 2.9, Rmk. 2.10], for Z and for each $V \in \mathrm{dAff}\,/Z$, the functors $\mathbb{V}_Z : \mathrm{Perf}(Z)^{\mathrm{op}} \to \mathrm{PerfLinSt}_Z$ and $\mathbb{V}_V : \mathrm{Perf}(V)^{\mathrm{op}} \to \mathrm{PerfLinSt}_V$ are equivalences of ∞-categories. Moreover, by unfolding the functoriality of PerfLinSt described above, for each $V \in \mathrm{dAff}\,/Z$, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{PerfLinSt}_Z \arrow[r] & \mathrm{PerfLinSt}_V \\
\mathrm{Perf}(Z) \arrow[r] \arrow[u, "\sim"] & \mathrm{Perf}(V) \arrow[u, "\sim"']
\end{tikzcd}$$

Therefore, the following diagram also commutes:

$$\begin{tikzcd}
\mathrm{PerfLinSt}_Z \arrow[r] & \lim\limits_{V \in \mathrm{dAff}\,/Z} \mathrm{PerfLinSt}_V \\
\mathrm{Perf}(Z) \arrow[r, "\sim"'] \arrow[u, "\sim"] & \lim\limits_{V \in \mathrm{dAff}\,/Z} \mathrm{Perf}(V) \arrow[u, "\sim"']
\end{tikzcd}$$

As such, the top functor is an equivalence, so we have proved that $\lim\limits_{V \in \mathrm{dAff}\,/Z} \mathrm{PerfLinSt}_V \simeq \mathrm{PerfLinSt}_Z$ as required. This concludes the proof. ☐

Therefore, one always has the following equivalence.

Corollary 5.3.17. *For $Z \to X$ a map of derived stacks which is locally of finite presentation, there is a canonical equivalence of ∞-categories:*

$$\mathcal{F}ol(Z/X) \simeq \mathrm{PerfLinSt}_Z\,/\widetilde{\mathcal{L}^{gr}}(Z/X) - \mathcal{H}$$

Moreover, when $p : Z \to X$ admits a relative contangent, we also obtain an explicit description of $\widetilde{\mathcal{L}^{gr}}(Z/X)$, and can identify the terminal foliation with $\mathbf{DR}(Z/X)$.

Proposition 5.3.18. *Let $p : Z \to X$ be any map of derived stacks. Then, $\widetilde{\mathcal{L}^{gr}}(Z/X)$ is relatively affine over Z. Moreover, when p is locally of finite presentation and admits a relative cotangent $\mathbb{L}_{Z/X}$, then there is a canonical equivalence $\widetilde{\mathcal{L}^{gr}}(Z/X) \simeq \mathrm{Spec}_Z(\mathbf{DR}(Z/X))$ as $\mathbb{G}_m$-equivariant derived stacks over Z, where $\mathbf{DR}(Z/X) := \mathrm{Sym}_{\mathcal{O}_Z}(\mathbb{L}_{Z/X}[1])$.*

Proof. Without assumptions on $p : Z \to X$, let us first prove that the canonical map $q : \widetilde{\mathcal{L}^{gr}}(Z/X) \to Z$ is relatively affine. Given a map $u : \mathrm{Spec}(A) \to X$, we can consider the following diagram:

$$\begin{tikzcd}
\widetilde{\mathcal{L}^{gr}}(Z_u/\operatorname{Spec}(A)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{L}^{gr}}(Z/X) \arrow[d] \\
Z_u \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d] \\
\operatorname{Spec}(A) \arrow[r, "u"'] & X
\end{tikzcd}$$

Notice that it suffices to check that all maps $\widetilde{\mathcal{L}^{gr}}(Z_u/\operatorname{Spec}(A)) \to Z_u$ are relatively affine to conclude that the map $q : \widetilde{\mathcal{L}^{gr}}(Z/X) \to Z$ is relatively affine. This is because, given any map $v : \operatorname{Spec}(B) \to Z$, we can then form the following diagram:

$$\begin{tikzcd}
v^*\widetilde{\mathcal{L}^{gr}}(Z/X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{L}^{gr}}(Z_u/\operatorname{Spec}(B)) \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & \widetilde{\mathcal{L}^{gr}}(Z/X) \arrow[d] \\
\operatorname{Spec}(B) \arrow[r] & Z_u \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & Z \arrow[d] \\
& \operatorname{Spec}(B) \arrow[r, "v"] \arrow[rr, bend right=20, "u"'] & Z \arrow[r, "p"] & X
\end{tikzcd}$$

Therefore, assuming $\widetilde{\mathcal{L}^{gr}}(Z_u/\operatorname{Spec}(B)) \to Z_u$ is relatively affine, we do obtain that $v^*\widetilde{\mathcal{L}^{gr}}(Z/X)$ is affine over $\operatorname{Spec}(B)$. We are thus reduced to proving the statement in the case where $X = \operatorname{Spec}(A)$ is affine. In this case, there is a canonical functor $\mathrm{dAff}/Z \to \mathrm{Arr}(\mathrm{dAff})/p$, which maps $v : \operatorname{Spec}(B) \to Z$ to the following commutative square in $\mathrm{Arr}(\mathrm{dAff})/p$:

$$\begin{tikzcd}
\operatorname{Spec}(B) \arrow[r, "v"] \arrow[d, "p\circ v"'] & Z \arrow[d, "p"] \\
\operatorname{Spec}(A) \arrow[r, equal] & \operatorname{Spec}(A)
\end{tikzcd}$$

Moreover, this functor is left adjoint to the forgetful functor $\mathrm{Arr}(\mathrm{dAff})/p \to \mathrm{dAff}/Z$ which only remembers the top map of each commutative square, so the above functor $\mathrm{dAff}/Z \to \mathrm{Arr}(\mathrm{dAff})/p$ is final. In particular, we obtain that:

$$\begin{aligned}
\widetilde{\mathcal{L}^{gr}}(Z/X) &:= \underset{(V\to U)\in \mathrm{Arr}(\mathrm{dAff})/p}{\operatorname{colim}} \mathcal{L}^{gr}(V/U) \\
&\simeq \underset{v:\operatorname{Spec}(B)\to Z}{\operatorname{colim}} \mathcal{L}^{gr}(B/A)
\end{aligned}$$

Consider now the family $(\mathcal{L}^{gr}(B/A))_{v:\operatorname{Spec}(B)\to Z}$, which defines an object in the ∞-category $\lim\limits_{v:\operatorname{Spec}(B)\to Z} \mathrm{dSt}/\operatorname{Spec}(B)$. The latter ∞-category is canonically equivalent to dSt/Z: this equivalence is given by taking the colimit of the family, and its inverse is given by mapping a derived stack $T \to Z$ to the family $(v^*T)_{v:\operatorname{Spec}(B)\to Z}$. Applying this description of the equivalence to the above family, we obtain that the derived stack over Z associated to this family recovers exactly $\widetilde{\mathcal{L}^{gr}}(Z/X)$ by the above computation, and that given any $v : \mathrm{Spec}(B) \to Z$ we have $v^*\widetilde{\mathcal{L}^{gr}}(Z/X) \simeq \mathcal{L}^{gr}(B/A) \simeq \mathrm{Spec}_B(\mathbf{DR}(B/A))$. We thus obtain that $\widetilde{\mathcal{L}^{gr}}(Z/X)$ is relatively affine over Z.

Therefore, we know that $\widetilde{\mathcal{L}^{gr}}(Z/X) \simeq \mathrm{Spec}_Z(C)$ for some $C \in \mathrm{CAlg}(\mathrm{QCoh}(Z))$, and moreover that $C \simeq q_*\widetilde{\mathcal{L}^{gr}}(Z/X)$. In fact, we can write this as $\mathcal{O}_{/Z}(\widetilde{\mathcal{L}^{gr}}(Z/X))$ in the sense of reminder 5.1.3, and since the functor $\mathcal{O}_{/Z}$ is left adjoint to Spec_Z, it sends colimits to limits. Therefore, in general, we obtain that

$$C \simeq \lim_{\phi:(\mathrm{Spec}(B)\to\mathrm{Spec}(A))\to p} v(\phi)_*\mathbf{DR}(B/A)$$

where $v(\phi) : \mathrm{Spec}(B) \to Z$ denotes the top arrow of each square $\phi \in \mathrm{Arr}(\mathrm{dAff})/p$.

We will now assume that $p : Z \to X$ is locally of finite presentation and admits a relative cotangent $\mathbb{L}_{Z/X} \in \mathrm{QCoh}(Z)$, and we can therefore define $\mathbf{DR}(Z/X) = \mathrm{Sym}_{\mathcal{O}_Z}(\mathbb{L}_{Z/X}[1]) \in \mathrm{CAlg}(\mathrm{QCoh}(Z))$. Note that by local finite presentation, $\mathbb{L}_{Z/X}$ is moreover perfect. We will now prove that $C \simeq \mathbf{DR}(Z/X)$, which will be similar to the proof of [CS26, Thm. 2.7]. Let $u : \mathrm{Spec}(A) \to X$, and consider the following diagram:

$$\begin{tikzcd}
Z_u \arrow[r, "u'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "p"] \\
\mathrm{Spec}(A) \arrow[r, "u"'] & X
\end{tikzcd}$$

Then, by [CHS25, Cor. B.5.6], we obtain that:

$$\begin{aligned}
\mathbf{DR}(Z/X) &\simeq \lim_{u:\mathrm{Spec}(A)\to X} u'_*u'^*\mathbf{DR}(Z/X) \\
&\simeq \lim_{u:\mathrm{Spec}(A)\to X} u'_*\mathbf{DR}(Z_u/\mathrm{Spec}(A))
\end{aligned}$$

where the identification $u'^*\mathbf{DR}(Z/X) \simeq \mathbf{DR}(Z_u/\mathrm{Spec}(A))$ comes from the fact that the square is a pull-back. More precisely, as a pull-back of p, p' is locally of finite presentation and admits a relative cotangent which is equivalent to $u'^*\mathbb{L}_{Z/X}$. We will now show that:

$$\mathbf{DR}(Z_u/\mathrm{Spec}(A)) \simeq \lim_{v:\mathrm{Spec}(B)\to Z_u} v_*\mathbf{DR}(B/A)$$

Assuming this is the case, we can end the proof by the following computation:

$$\begin{aligned}
\mathbf{DR}(Z/X) &\simeq \lim_{u:\mathrm{Spec}(A)\to X} u'_*\mathbf{DR}(Z_u/\mathrm{Spec}(A)) \\
&\simeq \lim_{u:\mathrm{Spec}(A)\to X} u'_* \lim_{v:\mathrm{Spec}(B)\to Z_u} v_*\mathbf{DR}(B/A) \\
&\simeq \lim_{u:\mathrm{Spec}(A)\to X} \lim_{v:\mathrm{Spec}(B)\to Z_u} u'_*v_*\mathbf{DR}(B/A) \\
&\simeq \lim_{\phi:(\mathrm{Spec}(B)\to\mathrm{Spec}(A))\to p} v(\phi)_*\mathbf{DR}(B/A)
\end{aligned}$$

$$\simeq C$$

Let us denote $Z' := Z_u$ and $X' := \operatorname{Spec}(A)$. In this case, since X' is a derived affine scheme, we can apply [CS26, Cor. 1.29] to obtain in our notations that $\mathbb{V}(\mathbb{L}_{Z'/X'}[1]) \simeq \underset{v':\operatorname{Spec}(B)\to Z'}{\operatorname{colim}} \mathbb{V}(\mathbb{L}_{B/A}[1])$ as $\mathbb{G}_m$-equivariant stacks over Z. Therefore, taking graded functions $\mathcal{O}_{gr}$ and using the fact that it sends colimits to limits, we obtain that $\mathcal{O}_{gr}(\mathbb{V}(\mathbb{L}_{Z'/X'}[1])) \simeq \underset{v':\operatorname{Spec}(B)\to Z'}{\lim} \mathcal{O}_{gr}(\mathbb{V}(\mathbb{L}_{B/A}[1]))$. Using the fact that $p : Z \to X$ is locally of finite presentation, we obtain that $p' : Z' \to X'$ is also locally of finite presentation. As a result, we obtain that $\mathbb{L}_{Z'/X'}$ is perfect and that we can compute the same limit when restricting to affines $v' : \operatorname{Spec}(B) \to Z'$ such that the composite $\operatorname{Spec}(B) \to Z' \to \operatorname{Spec}(A)$ is of finite presentation and hence we can assume the $\mathbb{L}_{B/A}$ are perfect B-modules as well. As such, by [Mou21, Thm. 2.5], we can identify $\mathcal{O}_{gr}(\mathbb{V}(\mathbb{L}_{Z'/X'}[1])) \simeq \mathbf{DR}(Z'/X')$ as graded Z'-algebras and for any $v' : \operatorname{Spec}(B) \to Z'$ we have $\mathcal{O}_{gr}(\mathbb{V}(\mathbb{L}_{B/A}[1])) \simeq v'_*\mathbf{DR}(B/A)$. Combining the above equivalences, we obtain as required that $\mathbf{DR}(Z'/X') \simeq \underset{v':\operatorname{Spec}(B)\to Z'}{\lim} v'_*\mathbf{DR}(B/A)$ as graded Z'-algebras. This implies by the above computation both the fact that $\mathcal{O}_{/Z}(\widetilde{\mathcal{L}^{gr}}(Z/X)) \simeq \mathbf{DR}(Z/X)$ and that this identification holds as graded algebras. Together with the fact that $\widetilde{\mathcal{L}^{gr}}(Z/X)$ is relatively affine over Z, this implies that $\widetilde{\mathcal{L}^{gr}}(Z/X) \simeq \operatorname{Spec}_Z(\mathbf{DR}(Z/X))$ as $\mathbb{G}_m$-equivariant stacks over Z, and concludes the proof. ☐

Corollary 5.3.19. *Let $p : Z \to X$ be a map of derived stacks which is locally of finite presentation and admits a relative cotangent. Then, the terminal object of* $\mathrm{dSt}\,/\widetilde{\mathcal{L}^{gr}}(Z/X) - \mathcal{H}$ *is equivalent to* $\operatorname{Spec}_Z(\mathbf{DR}(Z/X))$ *as a $\mathbb{G}_m$-equivariant stack over Z, so it belongs to* $\mathrm{PerfLinSt}_Z\,/\widetilde{\mathcal{L}^{gr}}(Z/X) - \mathcal{H}$ *and is the terminal foliation.*

Notation 5.3.20. By abuse of notation, we will also denote by $\mathbf{DR}(Z/X)$ the terminal foliation in the above case.

Remark 5.3.21. Note that $\mathbf{DR}(Z/X)$ usually denotes a sheaf of commutative algebras over Z, so it is the data of a commutative B-algebra for each map $v : \operatorname{Spec}(B) \to Z$, with transition maps between their usual pull-backs as described in proposition 3.4.10. On the other hand, the data of a graded mixed algebra on Z relative to X is the data of an A-linear graded mixed B-algebra for each commutative square $(\operatorname{Spec}(B) \to \operatorname{Spec}(A)) \to p$, where $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ can be assumed to be of finite presentation as in definition 5.3.4 since $p : Z \to X$ is locally of finite presentation, and with transition maps between their pull-backs as graded mixed algebras, which are very different from the usual pull-backs as explained in construction 4.2.14. Therefore, it is not immediate to see how to associate the second kind of data from the first, and viewing $\mathbf{DR}(Z/X)$ as a graded mixed algebra on Z relative to X does require a clarification, which explains the need to prove the above results.

Moreover, note that even when $p : Z \to X$ does not admit a genuine relative cotangent, for formal reasons there always exists a terminal derived foliation on Z relative to X (more explicitly, to a commutative square $(\operatorname{Spec}(B) \to \operatorname{Spec}(A)) \to p$ where $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$

is of finite presentation, it associates $\mathbf{DR}(B/A)$, which is a derived foliation whose cotangent $\mathbb{L}_{B/A}$ is thus perfect). However, we expect its associated Z-algebra to be equivalent to $\mathrm{Sym}_{\mathcal{O}_Z}(\mathbb{L}_{Z/X}^{big,qcoh}[1])$, where $\mathbb{L}_{Z/X}^{big}$ the big $\mathcal{O}_Z$-module of derivations as defined in remark 3.5.11. We will thus avoid denoting it as $\mathbf{DR}(Z/X)$ outside of the above case.

5.3.3 Equivalence for Relative Derived Deligne-Mumford Stacks

We will now compare $\widetilde{\mathcal{L}^{gr}}$ to $\mathcal{L}^{gr}$, with the final goal of proving that they coincide when $Z \to X$ is a relative derived Deligne-Mumford stack locally of finite presentation in theorem 5.3.24, which combined with corollary 5.3.17 provides a more explicit description of $\mathcal{F}ol(Z/X)$ in this case. To begin with, we will inspect properties of $\mathcal{L}^{gr}$ and $\widetilde{\mathcal{L}^{gr}}$.

More precisely, when $Z \to X$ is a relative derived Deligne-Mumford stack, we will want to show that the canonical map $\widetilde{\mathcal{L}^{gr}}(Z/X) \to \mathcal{L}^{gr}(Z/X)$ is an equivalence by reducing to the case where X is affine and Z is a relative derived Deligne-Mumford stack over X. The next two propositions will essentially enable us to do so.

Proposition 5.3.22. *Let $Z \to X$ be any map of derived stacks, and $u : U \to X$ any map of derived stacks. Then $u^*\mathcal{L}^{gr}(Z/X) \simeq \mathcal{L}^{gr}(u^*Z/U)$ as derived stacks over U.*

Proof. Let $T \to U \in \mathrm{dSt}/U$, we have the following chain of equivalences:

$$\begin{aligned}
\mathbf{Map}_U(T, u^*\mathcal{L}^{gr}(Z/X)) &= \mathbf{Map}_U(T, u^*\underline{\mathbf{Map}}_X(B\mathbb{G}_a \times X, Z)) \\
&\simeq \mathbf{Map}_X(u_!T, \underline{\mathbf{Map}}_X(B\mathbb{G}_a \times X, Z)) \\
&\simeq \mathbf{Map}_X((u_!T) \times_X (B\mathbb{G}_a \times X), Z) \\
&\simeq \mathbf{Map}_X(u_!(T \times_U (B\mathbb{G}_a \times U)), Z) \\
&\simeq \mathbf{Map}_U(T \times_U (B\mathbb{G}_a \times U), u^*Z) \\
&\simeq \mathbf{Map}_U(T, \underline{\mathbf{Map}}_U(B\mathbb{G}_a \times U, u^*Z)) \\
&= \mathbf{Map}_U(T, \mathcal{L}^{gr}(u^*Z/U))
\end{aligned}$$

where the equivalence $(u_!T) \times_X (B\mathbb{G}_a \times X) \simeq u_!(T \times_U (B\mathbb{G}_a \times U))$ can be seen in the following diagram:

$$\begin{tikzcd}
 & & Z \arrow[d] & \\
T \arrow[r] \arrow[d] & U \arrow[r, "u"] & X \arrow[r] & \bullet \\
T \times_X (X \times B\mathbb{G}_a) \arrow[r] \arrow[ur, phantom, "\urcorner", very near start] & U \times_X (X \times B\mathbb{G}_a) \arrow[r] \arrow[u] \arrow[ur, phantom, "\urcorner", very near start] & X \times B\mathbb{G}_a \arrow[r] \arrow[u] \arrow[ur, phantom, "\urcorner", very near start] & B\mathbb{G}_a \arrow[u] \\
T \times_U (U \times B\mathbb{G}_a) \arrow[r] \arrow[u, dash, "\sim"] \arrow[ur, phantom, "\urcorner", very near start] & U \times B\mathbb{G}_a \arrow[u, dash, "\sim"'] & &
\end{tikzcd}$$

Being natural in T, this equivalence proves that $u^*\mathcal{L}^{gr}(Z/X) \simeq \mathcal{L}^{gr}(u^*Z/U)$. Note that this equivalence is also $\mathcal{H}$-equivariant, although we will not need this fact. ☐

A similar statement for $\widetilde{\mathcal{L}^{gr}}$ can be obtained, compatibly with the above statement:

Proposition 5.3.23. *Let $p : Z \to X$ be a map of arbitrary derived stacks, and $u : U \to X$ be a map of arbitrary derived stacks. Then, $u^*\widetilde{\mathcal{L}^{gr}}(Z/X) \simeq \widetilde{\mathcal{L}^{gr}}(u^*Z/U)$. Moreover, the following diagram commutes:*

$$\begin{tikzcd}
\widetilde{\mathcal{L}^{gr}}(u^*Z/U) \arrow[r] \arrow[d, dash, "\simeq" sloped] & \mathcal{L}^{gr}(u^*Z/U) \arrow[d, dash, "\simeq" sloped] \\
u^*\widetilde{\mathcal{L}^{gr}}(Z/X) \arrow[r] & u^*\mathcal{L}^{gr}(Z/X)
\end{tikzcd}$$

Proof. Consider the category of commutative diagrams of the following form:

$$\begin{tikzcd}
U' \arrow[r] \arrow[d] & X' \arrow[d] & Z' \arrow[l] \arrow[d] \\
U \arrow[r] & X & Z \arrow[l]
\end{tikzcd}$$

where U', X', and Z' are affines. We will denote this category by $\mathrm{dAff}\,/Cospan(U, X, Z)$. For simplicity, we will denote its objects as triples (U', X', Z') and omit the maps and their commutativity in the notation.

We know that the colimit of $U' \to X' \leftarrow Z'$ over this category is equivalent to $U \to X \leftarrow Z$. This is essentially because such objects generate the topos $Cospan(\mathrm{dSt})/(U \to X \leftarrow Z)$, which in turn is implied by [HTT, Prop. 6.3.4.9]. Consider an object of this category, we can form the following commutative diagram:

$$\begin{tikzcd}
& u'^*Z' \arrow[rrr] \arrow[ddl] \arrow[d, dashed] \arrow[drrr, phantom, "\lrcorner", very near start] & & & Z' \arrow[d] \arrow[ddl] \\
& U' \arrow[rrr, dashed, "u'"] \arrow[ddl, dashed] & & & X' \arrow[ddl] \\
u^*Z \arrow[rrr] \arrow[d] \arrow[drrr, phantom, "\lrcorner", very near start] & & & Z \arrow[d] & \\
U \arrow[rrr, "u"] & & & X &
\end{tikzcd}$$

Note that u'^*Z' is also affine since U', X' and Z' are affines. We also know that the colimit of the pull-back square formed by u'^*Z', Z', U' and X' is the pull-back square formed by u^*Z, Z, U and X. By corollary 5.3.12, this implies that:

$$\widetilde{\mathcal{L}^{gr}}(u^*Z/U) \simeq \underset{(U',X',Z) \in \mathrm{dAff}\,/Cospan(U,X,Z)}{\operatorname{colim}} \widetilde{\mathcal{L}^{gr}}(u'^*Z'/U')$$

and also:

$$\widetilde{\mathcal{L}^{gr}}(Z/X) \simeq \operatorname*{colim}_{(U',X',Z)\in \mathrm{dAff}\,/Cospan(U,X,Z)} \widetilde{\mathcal{L}^{gr}}(Z'/X')$$

Moreover, for (U', X', Z') such an object, since u'^*Z' is also affine, we also know that $\widetilde{\mathcal{L}^{gr}}(u'^*Z'/U') \simeq \mathcal{L}^{gr}(u'^*Z'/U')$ by corollary 5.3.12, and that $\mathcal{L}^{gr}(u'^*Z'/U') \simeq u'^*L^{gr}(Z'/X')$ by proposition 5.3.22, and finally $u'^*L^{gr}(Z'/X') \simeq u'^*\widetilde{\mathcal{L}^{gr}}(Z'/X')$ again by corollary 5.3.12. Therefore, we can compute:

$$\begin{aligned}
\widetilde{\mathcal{L}^{gr}}(u^*Z/U) &\simeq \operatorname*{colim}_{(U',X',Z)\in \mathrm{dAff}\,/Cospan(U,X,Z)} \widetilde{\mathcal{L}^{gr}}(u'^*Z'/U') \\
&\simeq \operatorname*{colim}_{(U',X',Z)\in \mathrm{dAff}\,/Cospan(U,X,Z)} u'^*\widetilde{\mathcal{L}^{gr}}(Z'/X') \\
&= \operatorname*{colim}_{(U',X',Z)\in \mathrm{dAff}\,/Cospan(U,X,Z)} U' \times_{X'} \widetilde{\mathcal{L}^{gr}}(Z'/X')
\end{aligned}$$

Now, since colimits commute with finite limits and in particular with pull-backs, we know that this last colimit is equivalent to the pull-back of the following objects:

$$\begin{aligned}
U &\simeq \operatorname*{colim}_{(U',X',Z)\in \mathrm{dAff}\,/Cospan(U,X,Z)} U' \\
X &\simeq \operatorname*{colim}_{(U',X',Z)\in \mathrm{dAff}\,/Cospan(U,X,Z)} X' \\
\widetilde{\mathcal{L}^{gr}}(Z/X) &\simeq \operatorname*{colim}_{(U',X',Z)\in \mathrm{dAff}\,/Cospan(U,X,Z)} \widetilde{\mathcal{L}^{gr}}(Z'/X')
\end{aligned}$$

where the last equivalence holds because by corollary 5.3.12 since $Z \to X$ is the colimit of the $Z' \to X'$ over this category. Therefore, we obtain that:

$$\begin{aligned}
\widetilde{\mathcal{L}^{gr}}(u^*Z/U) &\simeq \operatorname*{colim}_{(U',X',Z)\in \mathrm{dAff}\,/Cospan(U,X,Z)} U' \times_{X'} \widetilde{\mathcal{L}^{gr}}(Z'/X') \\
&\simeq U \times_X \widetilde{\mathcal{L}^{gr}}(Z/X) \\
&= u^*\widetilde{\mathcal{L}^{gr}}(Z/X)
\end{aligned}$$

So we obtain the desired equivalence.

Moreover, by functoriality of $\mathcal{L}^{gr}$, we know that for each object $(U', X', Z') \in \mathrm{dAff}\,/Cospan(U, X, Z)$, the following square is commutative:

$$\begin{tikzcd}
\mathcal{L}^{gr}(Z'/X') \arrow[r] & \mathcal{L}^{gr}(Z/X) \\
\mathcal{L}^{gr}(u'^*Z'/U') \arrow[r] \arrow[u] \arrow[d, dash, "\sim"'] & \mathcal{L}^{gr}(u^*Z/U) \arrow[u] \arrow[d, dash, "\sim"] \\
u'^*\mathcal{L}^{gr}(Z'/U') \arrow[r] & u^*\mathcal{L}^{gr}(u^*Z/U)
\end{tikzcd}$$

Passing to the colimit over dAff $/Cospan(U, X, Z)$, we obtain that the following square is commutative:

$$\begin{tikzcd}
\widetilde{\mathcal{L}^{gr}}(Z/X) \arrow[r] & \mathcal{L}^{gr}(Z/X) \\
\widetilde{\mathcal{L}^{gr}}(u^*Z/X) \arrow[r] \arrow[u] \arrow[d, dash, "\sim"'] & \mathcal{L}^{gr}(u^*Z/U) \arrow[u] \arrow[d, dash, "\sim"] \\
u^*\widetilde{\mathcal{L}^{gr}}(Z'/U') \arrow[r] & u^*\mathcal{L}^{gr}(u^*Z/U)
\end{tikzcd}$$

This proves the second statement of this proposition, and concludes the proof. ☐

To conclude this section, we will show the following theorem.

Theorem 5.3.24. *Assuming $p : Z \to X$ is a relative derived Deligne-Mumford stack, the canonical map:*

$$\widetilde{\mathcal{L}^{gr}}(Z/X) \to \mathcal{L}^{gr}(Z/X)$$

is an equivalence.

Moreover, since this canonical map is $\mathcal{H}$-equivariant definition 5.3.7, and is a map of derived stacks over Z, these stacks are equivalent as $\mathcal{H}$-equivariant stacks and as stacks over Z.

Proof. To check that $\widetilde{\mathcal{L}^{gr}}(Z/X) \to \mathcal{L}^{gr}(Z/X)$ is an equivalence, it suffices to check that the pull-back of this map along any map $u : U \to X$ is an equivalence, where U is affine. Let $u : U \to X$ be such a map, by proposition 5.3.23, we know that the pull-back of the above canonical map is exactly the canonical map $\widetilde{\mathcal{L}^{gr}}(u^*Z/U) \to \mathcal{L}^{gr}(u^*Z/U)$. Notice that by assumption, since $Z \to X$ is a relative derived Deligne-Mumford stack, u^*Z is now a derived Deligne-Mumford stack, and U is affine. Therefore, we are reduced to proving the statement in the case where X is affine and Z is a derived Deligne-Mumford stack. We will now assume that this is the case.

By definition 4.3.5, we have that:

$$\mathcal{L}^{gr}(Z/X) := \underline{\mathbf{Map}}_X(B\mathbb{G}_a \times X, Z)$$

Note that, as derived stacks over X, there is a canonical equivalence:

$$\underline{\mathbf{Map}}_X(B\mathbb{G}_a \times X, Z) \simeq \underline{\mathbf{Map}}(B\mathbb{G}_a, Z) \times_{\underline{\mathbf{Map}}(B\mathbb{G}_a, X)} X$$

where the map $u : \mathbf{Map}(B\mathbb{G}_a, Z) \to \mathbf{Map}(B\mathbb{G}_a, X)$ is induced by $p : Z \to X$ and the map $s : X \to \mathbf{Map}(B\mathbb{G}_a, X)$ is the transpose of the projection $B\mathbb{G}_a \times X \to X$. To see this, it suffices to show that for every $y : Y \to X$ in dSt $/X$, there is a natural equivalence

$$\mathbf{Map}_{\mathrm{dSt}\,/X}\Big(Y, \underline{\mathbf{Map}}_X(B\mathbb{G}_a \times X, Z)\Big) \simeq \mathbf{Map}_{\mathrm{dSt}\,/X}\Big(Y, \underline{\mathbf{Map}}(B\mathbb{G}_a, Z) \times_{\underline{\mathbf{Map}}(B\mathbb{G}_a, X)} X\Big).$$

Let $y : Y \to X$, we have:

$$\begin{aligned}\mathbf{Map}_{\mathrm{dSt}\,/X}\Big(Y, \underline{\mathbf{Map}}_X(B\mathbb{G}_a \times X, Z)\Big) &\simeq \mathbf{Map}_{\mathrm{dSt}\,/X}\Big(Y \times_X (B\mathbb{G}_a \times X), Z\Big) \\ &\simeq \mathbf{Map}_{\mathrm{dSt}\,/X}(Y \times B\mathbb{G}_a, Z)\end{aligned}$$

since $Y \times_X (B\mathbb{G}_a \times X) \simeq Y \times B\mathbb{G}_a$ as derived stacks over X, where the structural map $Y \times B\mathbb{G}_a \to X$ is the composite of the projection $pr_1 : Y \times B\mathbb{G}_a \to Y$ with $y : Y \to X$. Now, a morphism $f : Y \times B\mathbb{G}_a \to Z$ over X is precisely the data of a morphism $f : Y \times B\mathbb{G}_a \to Z$ in dSt together with a homotopy $p \circ f \simeq y \circ pr_1$. By adjunction, the data of f in dSt corresponds to a morphism $\widetilde{f} : Y \to \underline{\mathbf{Map}}(B\mathbb{G}_a, Z)$, and under this correspondence the data of an equivalence $p \circ f \simeq y \circ pr_1$ becomes the data of an equivalence $u \circ \widetilde{f} \simeq s \circ y : Y \to \underline{\mathbf{Map}}(B\mathbb{G}_a, X)$, where as above $s : X \to \underline{\mathbf{Map}}(B\mathbb{G}_a, X)$ is the adjoint of the projection $B\mathbb{G}_a \times X \to X$, and $u : \underline{\mathbf{Map}}(B\mathbb{G}_a, Z) \to \underline{\mathbf{Map}}(B\mathbb{G}_a, X)$ is induced by p. Since X is the terminal object of dSt $/X$, the only map $Y \to X$ in dSt $/X$ is y, and thus we obtain that:

$$\mathbf{Map}_{\mathrm{dSt}\,/X}(Y \times B\mathbb{G}_a, Z) \simeq \mathbf{Map}_{\mathrm{dSt}\,/X}\Big(Y, \underline{\mathbf{Map}}(B\mathbb{G}_a, Z) \times_{\underline{\mathbf{Map}}(B\mathbb{G}_a, X)} X\Big).$$

Combining the previous equivalences, in total we obtain:

$$\mathbf{Map}_{\mathrm{dSt}\,/X}\Big(Y, \underline{\mathbf{Map}}_X(B\mathbb{G}_a \times X, Z)\Big) \simeq \mathbf{Map}_{\mathrm{dSt}\,/X}\Big(Y, \underline{\mathbf{Map}}(B\mathbb{G}_a, Z) \times_{\underline{\mathbf{Map}}(B\mathbb{G}_a, X)} X\Big).$$

and this equivalence is natural in $y : Y \to X$. As a result, we obtain the desired equivalence of derived stacks over X:

$$\mathcal{L}^{gr}(Z/X) := \underline{\mathbf{Map}}_X(B\mathbb{G}_a \times X, Z) \simeq \underline{\mathbf{Map}}(B\mathbb{G}_a, Z) \times_{\underline{\mathbf{Map}}(B\mathbb{G}_a, X)} X$$

Since Z is now a derived Deligne-Mumford stack, there exists an étale hypercovering $v_\bullet : V_\bullet \to Z$ where the $V_\bullet$ are disjoint union of affines. In particular, since the colimit of the maps $(V_\bullet \to X)$ in Arr(dSt)$/p$ is the map $(Z \to X)$, by corollary 5.3.12 we know that:

$$\widetilde{\mathcal{L}^{gr}}(Z/X) \simeq \operatorname{colim} \widetilde{\mathcal{L}^{gr}}(V_\bullet/X) \simeq \operatorname{colim} \mathcal{L}^{gr}(V_\bullet/X)$$

where the last equivalence also holds by corollary 5.3.12 since the $V_\bullet$ are disjoint union of affines and X is affine. Note that by functoriality of $\mathcal{L}^{gr}$, there are canonical maps $\mathcal{L}^{gr}(V_\bullet/X) \to \mathcal{L}^{gr}(Z/X)$, which induces a canonical map colim $\mathcal{L}^{gr}(V_\bullet/X) \to \mathcal{L}^{gr}(Z/X)$. Moreover, the following triangle commutes by definition:

$$\begin{tikzcd}
\widetilde{\mathcal{L}^{gr}}(Z/X) \arrow[rr] & & \mathcal{L}^{gr}(Z/X) \\
& \operatorname{colim} \mathcal{L}^{gr}(V_\bullet/X) \arrow[ul, "\sim"] \arrow[ur] &
\end{tikzcd}$$

Therefore, it suffices to show that $\operatorname{colim} \mathcal{L}^{gr}(V_\bullet/X) \to \mathcal{L}^{gr}(Z/X)$ is an equivalence to finish the proof. By [FPSS25, Thm. 2.11], we know that $\operatorname{colim} \underline{\mathbf{Map}}(B\mathbb{G}_a, V_\bullet) \xrightarrow{\sim} \underline{\mathbf{Map}}(B\mathbb{G}_a, Z)$ is an equivalence. Therefore, by pull-back, the map:

$$(\operatorname{colim} \underline{\mathbf{Map}}(B\mathbb{G}_a, V_\bullet)) \times_{\underline{\mathbf{Map}}(B\mathbb{G}_a, X)} X \xrightarrow{\sim} \underline{\mathbf{Map}}(B\mathbb{G}_a, Z) \times_{\underline{\mathbf{Map}}(B\mathbb{G}_a, X)} X$$

is an equivalence. Moreover, since colimits commute with pull-backs, we obtain that:

$$\operatorname{colim} \left(\underline{\mathbf{Map}}(B\mathbb{G}_a, V_\bullet) \times_{\underline{\mathbf{Map}}(B\mathbb{G}_a, X)} X\right) \xrightarrow{\sim} \underline{\mathbf{Map}}(B\mathbb{G}_a, Z) \times_{\underline{\mathbf{Map}}(B\mathbb{G}_a, X)} X$$

is also an equivalence. Finally, as identified above, this is exactly the fact that the map:

$$\operatorname{colim} \mathcal{L}^{gr}(V_\bullet/X) \xrightarrow{\sim} \mathcal{L}^{gr}(Z/X)$$

is an equivalence. This concludes the proof. ☐

To finish this chapter, note that combining this result with corollary 5.3.17, we obtain the following result.

Corollary 5.3.25. *If $Z \to X$ is a relative derived Deligne-Mumford stack locally of finite presentation, then there is a canonical equivalence of ∞-categories:*

$$\mathcal{F}ol(Z/X) \simeq \mathrm{PerfLinSt}_Z /\mathcal{L}^{gr}(Z/X) - \mathcal{H}$$

Chapter 6

Pushforward of Derived Foliations

Let $Z \to X$ and $f : X \to Y$ be a two maps of derived stacks. Our goal in this chapter is to construct a functor $f_{*,\mathcal{F}ol} : \mathcal{F}ol(Z/X) \to \mathcal{F}ol(f_*Z/Y)$ under some assumptions on $Z \to X$ and f, and use it to obtain in particular a derived foliation on some mapping stacks when the target comes equipped with one. Our plan to construct $f_{*,\mathcal{F}ol} : \mathcal{F}ol(Z/X) \to \mathcal{F}ol(f_*Z/Y)$ is to use corollary 5.3.25, so we will later restrict to the case where $Z \to X$ is a relative derived Deligne-Mumford stack locally of finite presentation. We will consider that an object of $\mathcal{F}ol(Z/X)$ is an $\mathcal{H}$-equivariant stack $\mathbb{V}(E) \to \mathcal{L}^{gr}(Z/X)$ where $E \in \operatorname{Perf}(Z)$, construct its Weil restriction $f_*\mathbb{V}(E)$ as in reminder 5.1.4 and as recalled below, and identify it with a perfect linear stack on f_*Z, equipped with an $\mathcal{H}$-action and a canonical $\mathcal{H}$-equivariant map to $\mathcal{L}^{gr}(f_*Z/Y)$, and finally identify this data as a derived foliation on f_*Z relative to Y.

More concretely, recall from reminder 5.1.4 that there is a functor

$$f_* : \operatorname{dSt}/X \to \operatorname{dSt}/Y$$

called the push-forward, direct image, or Weil restriction along f. By definition, it is the right adjoint to the usual pull-back $f^* : \operatorname{dSt}/Y \to \operatorname{dSt}/X$, and therefore given $U \to X$, the object $f_*(U)$ in dSt/Y satisfies the following universal property for all $V \to Y$:

$$\mathbf{Map}_Y(V, f_*(U)) := \mathbf{Map}_X(V \times_Y X, U)$$

Now, notice that, by functoriality of f_*, the data $\mathbb{V}(E) \to \mathcal{L}^{gr}(Z/X) \to Z$ is mapped to $f_*\mathbb{V}(E) \to f_*\mathcal{L}^{gr}(Z/X) \to f_*Z$.

In the first section of this chapter, with no assumption on $Z \to X$, we will prove that $f_*\mathcal{L}^{gr}(Z/X) \simeq \mathcal{L}^{gr}(f_*Z/Y)$ as $\mathcal{H}$-equivariant derived stacks over f_*Z, and that $f_*\mathbb{V}(E)$ is perfect linear over f_*Z, when assuming that f is proper-schematic and local complete intersection. This consitutes a pushforward of perfect linear stacks, as established in corollary 6.1.12.

In the second section of this chapter, assuming that $f_*Z \to Y$ is a relative derived Artin stack, we will note that $f_*Z \to Y$ will moreover be a derived Deligne-Mumford stack locally of finite presentation when adding that f is moreover flat, thus by corollary 5.3.25

we have that $f_*\mathbb{V}(E)$ corresponds to a derived foliation over f_*Z relative to Y, which yields the functor $f_{*,\mathcal{F}ol}$ in theorem 6.2.2 (2.2.6). Finally, applying this theorem to example 2.2.7 will yield the main theorem 6.2.4 (2.2.1) on mapping stacks.

6.1 PUSHFORWARD OF PERFECT LINEAR STACKS

Let us now follow the steps described above to construct the push-forward of perfect linear stacks, starting with two remarks on the push-forward, i.e. the Weil restriction along f of stacks over Z, as introduced in reminder 5.1.4. In all of this section, $Z \to X$ will be an arbitrary map of derived stacks.

Definition 6.1.1. *Let $Z \to X$ be an arbitrary map of derived stacks, and $f : X \to Y$ an arbitrary map of derived stacks. We will denote by:*

$$f_*^Z : \mathrm{dSt}\,/Z \to \mathrm{dSt}\,/f_*Z$$

the functor mapping a derived stack $T \to Z$, viewed as a map in $\mathrm{dSt}\,/X$*, to the map $f_*T \to f_*Z$ in* $\mathrm{dSt}\,/Y$*, providing an object in* $\mathrm{dSt}\,/f_*Z$*.*

Remark 6.1.2. The preceding functor is right adjoint to the functor mapping a derived stack $t' : T' \to f_*Z$ to the composite $f^*T' \xrightarrow{f^*t'} f^*f_*Z \xrightarrow{ev} Z$, where the map we will call $ev : f^*f_*Z \to Z$ is the co-unit of the adjunction $f^* \dashv f_* : \mathrm{dSt}\,/Y \rightleftarrows \mathrm{dSt}\,/X$. This left adjoint will be denoted by:

$$f_Z^* : \mathrm{dSt}\,/f_*Z \to \mathrm{dSt}\,/Z$$

Corollary 6.1.3. *As a right adjoint, the functor $f_*^Z : \mathrm{dSt}\,/Z \to \mathrm{dSt}\,/f_*Z$ commutes with limits and thus has a canonical monoidal structure.*

After establishing these results on f_*^Z, the next step in our construction consists in providing a canonical equivalence between $f_*^Z\mathcal{L}^{gr}(Z/X)$ and $\mathcal{L}^{gr}(f_*Z/Y)$. Note that graded loop stacks come equipped with canonical $\mathcal{H}$-actions, and since f_*^Z is monoidal by the above corollary, $f_*^Z\mathcal{L}^{gr}(Z/X)$ also inherits an $\mathcal{H}$-action. Therefore, it makes sense to further prove that this canonical map is an equivalence of $\mathcal{H}$-equivariant stacks over f_*Z, which we will need later on for our construction.

Lemma 6.1.4. *There is a canonical equivalence $f_*^Z\mathcal{L}^{gr}(Z/X) \simeq \mathcal{L}^{gr}(f_*Z/Y)$ as $\mathcal{H}$-equivariant stacks over f_*Z.*

Proof. We will start by comparing their functors of points simply as stacks over Y, where $f_*^Z\mathcal{L}^{gr}(Z/X)$ simply becomes $f_*\mathcal{L}^{gr}(Z/X)$. Let $T \in \mathrm{dSt}\,/Y$, we thus have:

$$\begin{aligned}
\mathbf{Map}_Y(T, f_*\mathcal{L}^{gr}(Z/X)) &\simeq \mathbf{Map}_X(f^*T, \mathcal{L}^{gr}(Z/X)) \\
&= \mathbf{Map}_X(f^*T, \underline{\mathbf{Map}}_X(X \times B\mathbb{G}_a, Z)) \\
&\simeq \mathbf{Map}_X(f^*T \times_X (X \times B\mathbb{G}_a), Z) \\
&\simeq \mathbf{Map}_X(f^*T \times_X f^*(Y \times B\mathbb{G}_a), Z) \\
&\simeq \mathbf{Map}_X(f^*(T \times_Y (Y \times B\mathbb{G}_a)), Z) \\
&\simeq \mathbf{Map}_Y(T \times_Y (Y \times B\mathbb{G}_a), f_*Z) \\
&\simeq \mathbf{Map}_Y(T, \underline{\mathbf{Map}}_Y(Y \times B\mathbb{G}_a, f_*Z)) \\
&= \mathbf{Map}_Y(T, \mathcal{L}^{gr}(f_*Z/Y))
\end{aligned}$$

Moreover, this chain of equivalences is canonically functorial in T. This provides an equivalence $f_*^Z\mathcal{L}^{gr}(Z/X) \simeq \mathcal{L}^{gr}(f_*Z/Y)$ as derived stacks over Y, and moreover by unfolding the definition we can see that the following diagram commutes:

$$\begin{tikzcd}
f_*\mathcal{L}^{gr}(Z/X) \arrow[rr, "\sim"] \arrow[dr] & & \mathcal{L}^{gr}(f_*Z/Y) \arrow[dl] \\
& f_*Z &
\end{tikzcd}$$

So it is in fact an equivalence of derived stacks over f_*Z. It remains to check that this equivalence is $\mathcal{H}$-equivariant. Now, assume that T as above is moreover equipped with an $\mathcal{H}$-action, and let us fix some $\mathcal{H}$-equivariant map $t : T \to f_*\mathcal{L}^{gr}(Z/X)$. It then suffices to notice that each object appearing in the above chain of equivalences has a canonical $\mathcal{H}$-action, and that each step of this chain of equivalences of mapping spaces preserves $\mathcal{H}$-equivariance, but this follows solely from unfolding the definitions involved in each step. For instance, the first equivalence $\mathbf{Map}_Y(T, f_*\mathcal{L}^{gr}(Z/X)) \simeq \mathbf{Map}_X(f^*T, \mathcal{L}^{gr}(Z/X))$ takes $t : T \to f_*\mathcal{L}^{gr}(Z/X)$ and associates to it the composite $f^*T \xrightarrow{f^*t} f^*f_*\mathcal{L}^{gr}(Z/X) \xrightarrow{ev} \mathcal{L}^{gr}(Z/X)$. Then, it suffices to note that f^*t remains $\mathcal{H}$-equivariant since f^* is monoidal, and that $ev : f^*f_*\mathcal{L}^{gr}(Z/X) \to \mathcal{L}^{gr}(Z/X)$ is indeed $\mathcal{H}$-equivariant, so their composite remains $\mathcal{H}$-equivariant. A similar immediate reasoning applies in every step above, and thus in particular when $T = f_*\mathcal{L}^{gr}(Z/Y)$, and $t = Id$ (which is $\mathcal{H}$-equivariant), one finds that the resulting map obtained at the end of the chain of equivalences $f_*\mathcal{L}^{gr}(Z/Y) \to \mathcal{L}^{gr}(f_*Z/Y)$ thus remains $\mathcal{H}$-equivariant. Note that this map is by definition the equivalence $f_*^Z\mathcal{L}^{gr}(Z/X) \simeq \mathcal{L}^{gr}(f_*Z/Y)$ of derived stacks over Y constructed above, and therefore this equivalence is $\mathcal{H}$-equivariant, so these stacks are moreover equivalent as $\mathcal{H}$-equivariant derived stacks. In conclusion, since $f_*^Z\mathcal{L}^{gr}(Z/X)$ and $\mathcal{L}^{gr}(f_*Z/Y)$ are equivalent as derived stacks over f_*Z, and since this equivalence is $\mathcal{H}$-equivariant, this concludes the proof. ☐

The next step in our construction now consists in proving that, assuming at some point that f is proper-schematic and local complete intersection, and given $E \in \mathrm{Perf}(Z)$, the push-forward of $\mathbb{V}(E) \to Z \to X$ along f is still perfect linear over f_*Z. The main difficulty in this situation is that the universal property of f_* applies after forgetting the structural map $f_*\mathbb{V}(E) \to f_*Z$ along $f_*Z \to Y$, while the universal property of linear stacks depends on viewing them over f_*Z. Therefore, in order to prove our statement, we will first prove that f_*^Z actually coincides with another functor having a universal property over f_*Z. Later on, we will apply this description to perfect linear stacks while assuming that f is proper-schematic and local complete intersection to compute the functor of points of $f_*\mathbb{V}(E)$ over f_*Z and show that it also corresponds to a perfect linear stack. More precisely, consider the following commutative diagram:

$$\begin{tikzcd}
Z \arrow[dr] & f^*f_*Z \arrow[l, "ev"'] \arrow[r, "\pi"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & f_*Z \arrow[d] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

where $ev : f^*f_*Z \to Z$ is the co-unit of the adjunction $f^* \dashv f_*$.

Lemma 6.1.5. *In the above situation, the functor* $f_*^Z : \mathrm{dSt}\,/Z \to \mathrm{dSt}\,/f_*Z$ *coincides with the composition* $\pi_* \circ ev^*$*, where* $ev^* : \mathrm{dSt}\,/Z \to \mathrm{dSt}\,/f^*f_*Z$ *is the pull-back and* $\pi_* : \mathrm{dSt}\,/f^*f_*Z \to \mathrm{dSt}\,/f_*Z$ *is the Weil restriction, see reminder 5.1.4.*

Proof. Note that f_*^Z is right adjoint to f_Z^*, while $\pi_* \circ ev^*$ is right adjoint to $ev_! \circ \pi^*$, so it suffices to check that these left adjoints coincide to conclude. Inspecting the definition of f_Z^* from definition 6.1.1, we can immediately rewrite it as the composite $ev_! \circ f_{f^*f_*Z}^*$, where $f_{f^*f_*Z}^* : \mathrm{dSt}\,/f_*Z \to \mathrm{dSt}\,/f^*f_*Z$ is the functor mapping a stack $T \to f_*Z$ to $f^*T \to f^*f_*Z$ using the functoriality of f^*. Therefore, it actually only suffices to check that $f_{f^*f_*Z}^*$ coincides with π^*. Let thus $T \in \mathrm{dSt}\,/f_*Z$. Consider the following diagram:

$$\begin{tikzcd}
\pi^*T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & T \arrow[d] \\
f^*f_*Z \arrow[r, "\pi"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & f_*Z \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

As a pasting of two cartesian squares, the exterior square is also cartesian, which precisely means that $\pi^*T \simeq f^*T$ as derived stacks over f^*f_*Z, as required. ☐

Let us now introduce the following definition and lemmas.

Definition 6.1.6. *Let $\phi : U \to V$ be a proper-schematic and local complete intersection map of derived stacks, and $E \in \mathrm{Perf}(U)$.*[1] *Then, by [Toë12, Thm. 2.1], $\phi_* : \mathrm{QCoh}(U) \to \mathrm{QCoh}(V)$ preserves perfect objects, and thus for $E \in \mathrm{Perf}(U)$ we can set:*

$$\phi_+ E = \phi_*(E^\vee)^\vee \in \mathrm{Perf}(V),$$

where $-^\vee$ denotes the dual of a perfect sheaf.

Remark 6.1.7. Note that ϕ_+ canonically promotes to a covariant functor $\mathrm{Perf}(U) \to \mathrm{Perf}(V)$, which is moreover left adjoint to $\phi^* : \mathrm{Perf}(V) \to \mathrm{Perf}(U)$. As in [Toë12, Prop. 1.4], also note that ϕ_* satisfies the base-change formula for pull-back of affines $\mathrm{Spec}(A) \to V$ (more generally for quasi-compact quasi-separated derived schemes $X \to V$).

Lemma 6.1.8. *In the above setting, there is a canonical equivalence $\phi_* \mathbb{V}(E) \simeq \mathbb{V}(\phi_+ E)$ as derived stacks over V.*

Proof. We will compare their functors of points as derived stacks over V. Let $v : T \to V$ be any map of derived stacks, where T is affine. We can form the following pull-back diagram:

$$\begin{tikzcd}
\phi^* T \arrow[r, "p"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "v"] \\
U \arrow[r, "\phi"'] & V
\end{tikzcd}$$

Freely using the fact that pull-back functors of quasi-coherent sheaves are monoidal and thus commute with $-^\vee$, we then have:

$$\begin{aligned}
\mathbf{Map}_V(T, \phi_* \mathbb{V}(E)) &\simeq \mathbf{Map}_U(\phi^* T, \mathbb{V}(E)) \\
&= \mathrm{Hom}_{\mathrm{QCoh}(\phi^* T)}(u^* E, \mathcal{O}_{\phi^* T}) \\
&\simeq \mathrm{Hom}_{\mathrm{QCoh}(\phi^* T)}(\mathcal{O}_{\phi^* T}, u^*(E^\vee)) \\
&\simeq \mathrm{Hom}_{\mathrm{QCoh}(\phi^* T)}(p^* \mathcal{O}_T, u^*(E^\vee)) \\
&\simeq \mathrm{Hom}_{\mathrm{QCoh}(T)}(\mathcal{O}_T, p_* u^*(E^\vee)) \quad (*), \text{ marked for lemma 6.1.10} \\
&\simeq \mathrm{Hom}_{\mathrm{QCoh}(T)}(\mathcal{O}_T, v^* \phi_*(E^\vee)), \text{ using the base-change formula} \\
&\simeq \mathrm{Hom}_{\mathrm{QCoh}(T)}(v^*(\phi_+ E), \mathcal{O}_T) \\
&= \mathbf{Map}_V(T, \mathbb{V}(\phi_+ E))
\end{aligned}$$

Note that the base-change formula applies since T is affine. Moreover, this equivalence is natural in $T \to V$, and since it is true for all affines T, the objects $\phi_* \mathbb{V}(E)$ and $\mathbb{V}(\phi_+(E))$ have equivalent functors of points in dSt/V, which concludes the proof. ☐

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[1] Note that by proper-schematic we mean a map $U \to V$ such that the pull-back along any derived affine $\mathrm{Spec}(A) \to V$ is a proper derived scheme $U_A \to \mathrm{Spec}(A)$. In particular, ϕ is a assumed to be a relative derived scheme.

Remark 6.1.9. This statement can be interpreted in the context of reminder A.2.1, where conditions on a map were compared to certain canonical adjunction squares to be Beck-Chevalley. In this remark, on the side of derived stacks, only the adjunction $f_! \dashv f^*$ was considered, while on this side there is always a canonical triple $f_! \dashv f^* \dashv f_*$, as in reminder 5.1.4. This is because, in general, there is no corresponding triple on the side of quasi-coherent sheaves, and only $f^* \dashv f_*$ can be considered on that side. However, in the above situation, the functor ϕ_+ does fit in a similar triple $\phi_+ \dashv \phi^* \dashv \phi_*$, and the above statement can be interpreted as a proof that the appropriate extra canonical square that exists in this situation is in fact Beck-Chevalley too, at least when restricted to symmetric algebras. We will neither formally state nor prove this fact in this thesis as the above result suffices.

Lemma 6.1.10. *In the same context as above, the equivalence $\phi_*\mathbb{V}(E) \simeq \mathbb{V}(\phi_+E)$ is $\mathbb{G}_m$-equivariant.*

Proof. Let us first recall that the action of $\mathbb{G}_m$ on $\mathbb{V}(\phi_+E)$ is given in the following way: for an affine stack $v : \operatorname{Spec}(A) \to V$, an A-point of $\mathbb{G}_m$ is an automorphism $\psi : A \simeq A$, an A-point of $\mathbb{V}(\phi_+E)$ is a map from v^*E to A, and the action of the first on the second is the composition of these two maps. In the meantime, an A-point of $\phi_*\mathbb{V}(E)$ is given by a map m from u^*E to $\mathcal{O}_{\phi^* \operatorname{Spec}(A)}$ (where u is the pull-back of v along ϕ), and by definition, the action of $\psi \in \mathbb{G}_m(A)$ on m consists firstly in using ψ to define an automorphism $\Psi : \phi^* \operatorname{Spec}(A) \simeq \phi^* \operatorname{Spec}(A)$ by pulling-back $v \circ \operatorname{Spec}(\psi)$ along ϕ, using Ψ to produce an automorphism of $\mathcal{O}_{\phi^* \operatorname{Spec}(A)}$, and finally composing it with m. The definition of Ψ is illustrated in the following commutative diagram:

$$\begin{tikzcd}
\phi^* \operatorname{Spec}(A) \arrow[r, "p"] \arrow[d, "\Psi"', "\simeq"] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec}(A) \arrow[d, "\simeq"', "\operatorname{Spec}(\psi)"] \\
\phi^* \operatorname{Spec}(A) \arrow[r, "p"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec}(A) \arrow[d, "v"] \\
U \arrow[r, "\phi"'] & V
\end{tikzcd}$$

In particular, notice that Ψ can alternatively be defined as the pull-back of $\operatorname{Spec}(\psi)$ along p. Inspecting the construction of the equivalence $\phi_*\mathbb{V}(E) \simeq \mathbb{V}(\phi_+E)$ in lemma 6.1.8, note that all but one step either consist in a dualisation or in a composition with an equivalence on the opposite side to where $\mathbb{G}_m$ acts, both of which type of steps are $\mathbb{G}_m$-equivariant (either by functoriality or by associativity of the composition). The sole exception is the one step marked with a star (*). This step consists in the following equivalence, due to an adjunction:

$$\operatorname{Hom}_{\operatorname{QCoh}(\phi^* \operatorname{Spec}(A))}(p^*A, u^*(E^\vee)) \simeq \operatorname{Hom}_{A-Mod}(A, p_*u^*(E^\vee)).$$

At this point, after unfolding the equivalences, the $\mathbb{G}_m$-action on the right-hand side is given by pre-composing the dual of an automorphism $\psi : A \simeq A$, while on the left-hand side ψ is first pulled-back along p and then dualised before being pre-composed. Now the equivalence itself takes a map $A \to p_* u^*(E^\vee)$ to its pull-back $p^* A \to p^* p_* u^*(E^\vee)$ and then post-composes it with the co-unit $p^* p_* \implies Id$, so by functoriality of p^* and associativity of the composition with the co-unit, this equivalence indeed is compatible with the actions of $\mathbb{G}_m(A)$ on both sides. ☐

Collecting what was proven so far, we obtain the following corollary.

Corollary 6.1.11. *For $f : X \to Y$ a proper-schematic and local complete intersection map between derived stacks, $Z \to X$ an arbitrary map of derived stack, and $E \in \mathrm{Perf}(Z)$, there is a canonical equivalence of $\mathbb{G}_m$-equivariant stacks over $f_* Z$:*

$$f_*^Z \mathbb{V}(E) \simeq \mathbb{V}(\pi_+ ev^* E),$$

where $ev : f^* f_* Z \to Z$ is the co-unit of $f^* \dashv f_*$ at Z, and $\pi : f^* f_* Z \to f_* Z$ is the projection. In particular, f_*^Z sends perfect linear stacks over Z to perfect linear stacks over $f_* Z$.

Proof. As a consequence of lemma 6.1.5, $f_*^Z \mathbb{V}(E) \simeq \pi_* ev^* \mathbb{V}(E)$ as $\mathbb{G}_m$-equivariant stacks over $f_* Z$. Moreover, $ev^* \mathbb{V}(E) \simeq \mathbb{V}(ev^* E)$ as $\mathbb{G}_m$-equivariant stacks over $f^* f_* Z$, as in reminder 5.1.4. Since E is perfect and ev^* is monoidal, $ev^* E$ is also perfect, and since f is proper-schematic and local complete intersection and π is its pull-back, π is also proper-schematic and local complete intersection. We can thus apply lemma 6.1.10, and obtain an equivalence $\pi_* \mathbb{V}(ev^* E) \simeq \mathbb{V}(\pi_+ ev^* E)$ as $\mathbb{G}_m$-equivariant stacks over $f_* Z$. Combining these three equivalences, we obtain as required an equivalence $f_*^Z \mathbb{V}(E) \simeq \mathbb{V}(\pi_+ ev^* E)$ as $\mathbb{G}_m$-equivariant stacks over $f_* Z$. ☐

Finally, using the equivalence $f_*^Z \mathcal{L}^{gr}(Z/X) \simeq \mathcal{L}^{gr}(f_* Z/Y)$ of $\mathcal{H}$-equivariant stacks over $f_* Z$ provided by lemma 6.1.4, we obtain our main construction.

Corollary 6.1.12. *For $f : X \to Y$ a proper-schematic and local complete intersection map between derived stacks, and $Z \to X$ an arbitrary map of derived stack, the direct image functor $f_* : \mathrm{dSt}/X \to \mathrm{dSt}/Y$ induces a canonical functor:*

$$f_* : \mathrm{PerfLinSt}_Z / \mathcal{L}^{gr}(Z/X) - \mathcal{H} \to \mathrm{PerfLinSt}_{f_* Z} / \mathcal{L}^{gr}(f_* Z/Y) - \mathcal{H}$$

Proof. Let $F \in \mathrm{PerfLinSt}_Z / \mathcal{L}^{gr}(Z/X) - \mathcal{H}$. Firstly, we will equip $f_* F$ with an $\mathcal{H}$-action over $f_* Z$, an $\mathcal{H}$-equivariant map to $\mathcal{L}^{gr}(f_* Z/Y)$, and show that it is perfect linear over $f_* Z$. Using the monoidal structure of f_*^Z in the sense of corollary 6.1.3, we can equip $f_* F$ with an $\mathcal{H}$-action over $f_* Z$, inherited from the structural one on F over Z. Similarly, the structural $\mathcal{H}$-equivariant map $F \to \mathcal{L}^{gr}(Z/X)$ is mapped to an $\mathcal{H}$-equivariant map $f^Z_* F \to f^Z_* \mathcal{L}^{gr}(Z/X)$. Moreover, by lemma 6.1.4, there is a canonical equivalence $f^Z_* \mathcal{L}^{gr}(Z/X) \simeq \mathcal{L}^{gr}(f_* Z/Y)$ as $\mathcal{H}$-equivariant stacks over $f_* Z$, which composed with the previous map produces an $\mathcal{H}$-equivariant map $f^Z_* F \to \mathcal{L}^{gr}(f_* Z/Y)$. Finally, by corollary 6.1.11, $f^Z_* F$ is perfect linear over $f_* Z$. As a consequence we have constructed a functor:

$$f_* : \mathrm{PerfLinSt}_Z \,/\mathcal{L}^{gr}(Z/X) - \mathcal{H} \to \mathrm{PerfLinSt}_{f_* Z} \,/\mathcal{L}^{gr}(f_* Z/Y) - \mathcal{H}$$

□

6.2 Pushforward of Derived Foliations

At this point we want to apply corollary 5.3.25 to replace both sides of the above functor with $\mathcal{F}ol(Z/X)$ and $\mathcal{F}ol(f_* Z/Y)$ respectively to conclude. We will thus assume that $Z \to X$ is relative derived Deligne-Mumford stack locally of finite presentation to apply it on the source, but we will need the following lemma to show that the target also satisfies this condition.

Lemma 6.2.1. *Let $Z \to X$ be a relative derived Deligne-Mumford stack locally of finite presentation, and $f : X \to Y$ a proper-schematic, flat, and local complete intersection map of derived stacks. Assume that $f_* Z \to Y$ is a relative derived Artin stack locally of finite presentation. Then, $f_* Z \to Y$ is a relative derived Deligne-Mumford stack locally of finite presentation.*

Proof. Since $f_* Z \to Y$ is assumed to be a relative derived Artin stack and locally of finite presentation, for it to be a relative derived Deligne-Mumford stack locally of finite presentation, it suffices to show that its truncation $t_0(f_* Z) \to t_0(Y)$ is a relative (non-derived) Deligne-Mumford stack. Let us now consider the truncation:

$$\begin{tikzcd}
t_0(Z) \arrow[d] & t_0(f_* Z) \arrow[d] \\
t_0(X) \arrow[r, "t_0(f)"'] & t_0(Y)
\end{tikzcd}$$

Since $Z \to X$ is a relative derived Deligne-Mumford stack, $t_0(Z) \to t_0(X)$ is a relative (non-derived) Deligne-Mumford stack, and similarly $t_0(f)$ is a proper-schematic, flat, and local complete intersection map of (non-derived) stacks, so we can apply [Ols06, Thm. 1.5], and obtain that $t_0(f)_* t_0(Z) \to t_0(Y)$ is a relative (non-derived) Deligne-Mumford stack. Finally, since f is schematic and flat, let us now show that $t_0(f)_* t_0(Z)$ is equivalent to $t_0(f_* Z)$, which will prove that $f_* Z \to Z$ is a relative derived Deligne-Mumford stack.

Let Spec(A) be a non-derived affine scheme, we have the following chain of equivalences, natural in Spec(A):

$$\begin{aligned}
&\mathbf{Map}_{\mathrm{St}}(\operatorname{Spec}(A), t_0(f_*Z)) \\
&\simeq \mathbf{Map}_{\mathrm{dSt}}(\operatorname{Spec}(A), f_*Z) \\
&\simeq \mathbf{Map}_{\mathrm{dSt}}(f^* \operatorname{Spec}(A), Z) \\
&\simeq \mathbf{Map}_{\mathrm{St}}(t_0(f)^* \operatorname{Spec}(A), t_0(Z)) \qquad (*) \\
&\simeq \mathbf{Map}_{\mathrm{St}}(\operatorname{Spec}(A), t_0(f)_* t_0(Z))
\end{aligned}$$

where the equivalence $(*)$ holds since, as f is schematic, $f^* \operatorname{Spec}(A)$ is a derived scheme; moreover, as f is flat, this derived scheme is in fact non-derived and coincides with $t_0(f)^* \operatorname{Spec}(A)$; and finally maps from non-derived schemes to derived stacks factor through their truncations. ☐

Combining these results, we obtain our push-forward of derived foliations.

Theorem 6.2.2 (Theorem 2.2.6)**.** *Let $Z \to X$ be a map of derived stacks which is a relative derived Deligne-Mumford stack locally of finite presentation, and $f : X \to Y$ a proper-schematic, flat, and local complete intersection map of derived stacks. Assume that $f_*Z \to Y$ is a relative derived Artin stack locally of finite presentation. Then, there is a push-forward functor:*

$$f_{*,\mathcal{F}ol} : \mathcal{F}ol(Z/X) \to \mathcal{F}ol(f_*Z/Y)$$

Proof. Applying corollary 6.1.12, we have a functor:

$$f_* : \operatorname{PerfLinSt}_Z / \mathcal{L}^{gr}(Z/X) - \mathcal{H} \to \operatorname{PerfLinSt}_{f_*Z} / \mathcal{L}^{gr}(f_*Z/Y) - \mathcal{H}$$

Moreover, by corollary 5.3.25, we have that $\mathcal{F}ol(Z/X) \simeq \operatorname{PerfLinSt}_Z / \mathcal{L}^{gr}(Z/X) - \mathcal{H}$ since $Z \to X$ is a relative derived Deligne-Mumford stack locally of finite presentation. We also have that $\mathcal{F}ol(f_*Z/Y) \simeq \operatorname{PerfLinSt}_{f_*Z} / \mathcal{L}^{gr}(f_*Z/Y) - \mathcal{H}$ since $f_*Z \to X$ is also a relative derived Deligne-Mumford stack locally of finite presentation by lemma 6.2.1. Therefore, using these equivalences, we obtain the push-forward functor:

$$f_{*,\mathcal{F}ol} : \mathcal{F}ol(Z/X) \to \mathcal{F}ol(f_*Z/Y)$$

☐

Remark 6.2.3. Being a push-forward, we can wonder whether this functor is a right adjoint. More precisely, remember that this functor is essentially the application of $f_*^Z : \mathrm{dSt}/Z \to \mathrm{dSt}/f_*Z$, which is right adjoint to the functor $f_Z^* : \mathrm{dSt}/f_*Z \to \mathrm{dSt}/Z$, so we can ask if this left adjoint also preserves perfect linear stacks and $\mathcal{H}$-actions and defines a left adjoint to $f_{*,\mathcal{F}ol}$. Note that the pull-back of derived foliations as defined in construction 4.2.14 does not apply in this context and this would thus be a different construction, since there is no canonical map $Z \to f_*Z$ making a commutative square with the existing maps $Z \to X$, $X \to Y$ and $f_*Z \to Y$. However, we do not expect f_Z^* to preserve neither perfect linear stacks nor their $\mathcal{H}$-actions in general or under reasonable conditions. This is because, by lemma 6.1.5 it can be rewritten as $ev_! \circ \pi^*$, where $\pi : f^*f_*Z \to f_*Z$ is the projection, and $ev : f^*f_*Z \to Z$ is the evaluation map (the co-unit of the adjunction $f^* \dashv f_*$). While π^* always preserves this structure, we do not expect $ev_!$ to preserve it under reasonsable conditions.

As a final application, we will now prove the main theorem 2.2.1.

Theorem 6.2.4 (Theorem 2.2.1)**.** *Let S be a base derived stack in characteristic 0. Let X be a proper-schematic, flat, and local complete intersection derived stack over S, and Y be a relative derived Deligne-Mumford stack locally of finite presentation over S, equipped with a derived foliation $\mathcal{F}$ relative to S. Then, assuming the derived mapping stack $\underline{\mathbf{Map}}_S(X, Y)$ is a relative derived Artin stack locally of finite presentation over S, it inherits a derived foliation $\mathcal{F}_{\mathbf{Map}}$ relative to S.*

Proof. Let us denote by $f : X \to S$ the structural map of X over S, which by assumption is proper-schematic, flat, and local complete intersection. Consider the derived stack $Z = X \times_S Y$ over X, and note that since Y is a relative derived Deligne-Mumford stack locally of finite presentation over S, Z is a relative derived Deligne-Mumford stack locally of finite presentation over X. Now recall that $f_*Z \simeq \underline{\mathbf{Map}}_S(X, Y)$ as derived stacks over S. To see this, let us compute its functor of points. For $\mathrm{Spec}(A) \to S$, we have the following equivalences, natural in $\mathrm{Spec}(A) \to S$:

$$\begin{aligned}
\mathbf{Map}_{\mathrm{dSt}\,/S}(\mathrm{Spec}(A), f_*Z) &\simeq \mathbf{Map}_{\mathrm{dSt}\,/X}(f^*\,\mathrm{Spec}(A), Z)\\
&= \mathbf{Map}_{\mathrm{dSt}\,/X}(f^*\,\mathrm{Spec}(A), X \times_S Y)\\
&= \mathbf{Map}_{\mathrm{dSt}\,/X}(f^*\,\mathrm{Spec}(A), f^*Y)\\
&\simeq \mathbf{Map}_{\mathrm{dSt}\,/S}(f_!f^*\,\mathrm{Spec}(A), Y)\\
&= \mathbf{Map}_{\mathrm{dSt}\,/S}(\mathrm{Spec}(A) \times_S X, Y)\\
&\simeq \mathbf{Map}_{\mathrm{dSt}\,/S}(\mathrm{Spec}(A), \underline{\mathbf{Map}}_S(X, Y))
\end{aligned}$$

As the functors of point coincide, $f_*Z \simeq \underline{\mathbf{Map}}_S(X, Y)$ as derived stacks over S. Therefore, the assumption that $\underline{\mathbf{Map}}_S(X, Y)$ is a relative derived Artin stack locally of finite presentation is exactly the assumption that f_*Z is a relative derived Artin stack locally of finite presentation. As such, the conditions of theorem 2.2.6 are satisfied, and there is a canonical push-forward functor $f_{*,\mathcal{F}ol} : \mathcal{F}ol(Z/X) \to \mathcal{F}ol(f_*Z/S) = \mathcal{F}ol(\underline{\mathbf{Map}}_S(X, Y)/S)$. Let now $\mathcal{F} \in \mathcal{F}ol(Y/S)$ denote the derived foliation that Y comes equipped with, and finally we denote by $p_Y : X \times_S Y \to Y$ the projection on Y. Note that it forms a commutative square with $f : X \to S$. We will denote by $\phi : (X \times_S X = Z \to X) \to (Y \to S)$ this commutative square. Since both $Y \to S$ and $Z \to X$ have perfect cotangents as they are locally of finite presentation, this induces a canonical pull-back functor $\phi^* : \mathcal{F}ol(Y/S) \to \mathcal{F}ol(Z/X)$, by a computation similar to remark 4.2.19. Therefore, we obtain a derived foliation on $\underline{\mathbf{Map}}_S(X, Y)$ by setting:

$$\mathcal{F}_{\mathbf{Map}} := f_{*,\mathcal{F}ol}(p_Y^* \mathcal{F}) \in \mathcal{F}ol(\underline{\mathbf{Map}}_S(X, Y)/S) \qquad \square$$

We will also prove that the induced derived foliation satisfies the property mentioned in the introduction.

Proposition 6.2.5 (Proposition 2.2.3)**.** *In the context of theorem 2.2.1 (6.2.4), let $\mathbb{T}_{\mathcal{F}}$ (resp. $\mathbb{T}_{\mathcal{F},\mathbf{Map}}$) denote the tangent complex of $\mathcal{F}$ (resp. $\mathcal{F}_{\mathbf{Map}}$), and $\mathbb{T}_{X/S}$ the tangent complex of X relative to S. For any S-point of $\underline{\mathbf{Map}}_S(X, Y)$, which corresponds to an S-linear map $g : X \to Y$, there is a canonical equivalence:*

$$\mathbb{T}_g \mathcal{F}_{\mathbf{Map}} \simeq \Gamma_S(X, g^* \mathbb{T}_{\mathcal{F}})$$

where $\Gamma_S : \mathrm{QCoh}(X) \to \mathrm{QCoh}(S)$ is the global-sections functor relative to the base S, i.e. the push-forward of quasi-coherent sheaves along the structural map $X \to S$, which recovers the usual global sections when $S = \mathrm{Spec}(k)$. More generally, the tangent complex of $\mathcal{F}_{\mathbf{Map}}$ itself is given by:

$$\mathbb{T}_{\mathcal{F},\mathbf{Map}} \simeq \pi_* ev^* \mathbb{T}_{\mathcal{F}}$$

where $ev : X \times_S \underline{\mathbf{Map}}_S(X, Y) \to Y$ is the evaluation, and $\pi : X \times_S \underline{\mathbf{Map}}_S(X, Y) \to \underline{\mathbf{Map}}_S(X, Y)$ is the projection

Proof. We will first prove the following equivalence:

$$\mathbb{T}_{\mathcal{F},\mathbf{Map}} \simeq \pi_* ev^* \mathbb{T}_{\mathcal{F}}$$

Note that by definition in theorem 2.2.1, we have:

$$\mathcal{F}_{\mathbf{Map}} := f_{*,\mathcal{F}ol}(p_Y^* \mathcal{F})$$

Moreover, by remark 4.2.19, if $\mathbb{L}_{\mathcal{F}}$ is the cotangent of $\mathcal{F}$, the cotangent of $p_Y^* \mathcal{F}$ is given by:

$$\mathbb{L}_{p_Y^* \mathcal{F}} \simeq \mathbb{L}_{X \times_S Y/X} \oplus_{p_Y^* \mathbb{L}_{Y/S}} p_Y^* \mathbb{L}_{\mathcal{F}}$$

Note that $\mathbb{L}_{X \times_S Y/X} \simeq p_Y^* \mathbb{L}_{Y/S}$, since $\mathbb{L}_{X \times_S Y/S} \simeq p_Y^* \mathbb{L}_{Y/S} \oplus p_X^* L_{X/S}$, where $p_X : X \times_S Y \to X$ is the projection, and $\mathbb{L}_{X \times_S Y/X}$ is defined as the quotient of $\mathbb{L}_{X \times_S Y/S}$ by $p_X^* \mathbb{L}_{X/S}$. As such, we obtain that:

$$\mathbb{L}_{p_Y^* \mathcal{F}} \simeq p_Y^* \mathbb{L}_{\mathcal{F}}$$

As noted in remark 4.2.19, in this case we would also obtain that $\mathbb{L}_{p_Y^*\mathcal{F}}$ happens to remain perfect since p_Y^* preserves perfects, without the need for further assumptions. More generally, pull-back of non-perfect derived foliations on cartesian square will always satisfy this relation for their cotangent, and thus preserve perfect derived foliations in these cases.

Applying the formula for the cotangent of the derived foliation induced by $f_{*,\mathcal{F}ol}$, which is provided by corollary 6.1.11, we obtain that:

$$\mathbb{L}_{\mathcal{F},\mathbf{Map}} \simeq \pi_*(ev'^* p_Y^* \mathbb{L}_{\mathcal{F}}^\vee)^\vee$$

where $ev' : X \times_S \underline{\mathbf{Map}}_S(X,Y) \to X \times_S Y$ is the co-unit of the adjunction $f^* \dashv f_*$ at $Z = X \times_S Y$. Note that $p_Y \circ ev' \simeq ev$, so that $ev'^* \circ p_Y^* \simeq ev^*$. Therefore, after dualizing, we do obtain the expected formula:

$$\mathbb{T}_{\mathcal{F},\mathbf{Map}} \simeq \pi_* ev^* \mathbb{T}_{\mathcal{F}}$$

We will now show that this formula induces the one at an S-point of $\underline{\mathbf{Map}}_S(X,Y)$. Let $u : S \to \underline{\mathbf{Map}}_S(X,Y)$ be an S-linear map, corresponding to an S-linear map $g : X \to Y$, as can be seen in the following commutative diagram:

$$\begin{tikzcd}
 & X \arrow[r, "f"] \arrow[d, "u'"] \arrow[dl, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "u"] \\
Y & X \times_S \underline{\mathbf{Map}}_S(X,Y) \arrow[l, "ev"] \arrow[r, "\pi"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathbf{Map}}_S(X,Y) \arrow[d] \\
 & X \arrow[r, "f"'] & S
\end{tikzcd}$$

Note that the square at the bottom right is a pull-back by definition, and the outer square is a pull-back since the compositions $S \to \underline{\mathbf{Map}}_S(X,Y) \to S$ and $X \to X \times_S \underline{\mathbf{Map}}_S(X,Y) \to X$ are the identity maps on S and X respectively, so by pasting the top square is a pull-back too. Also, by definition we have that $\mathbb{T}_g\mathcal{F}_{\mathbf{Map}} := u^*\mathbb{T}_{\mathcal{F},\mathbf{Map}}$, so by the previous equivalence we have:

$$\mathbb{T}_g\mathcal{F}_{\mathbf{Map}} \simeq u^*\pi_* ev^* \mathbb{T}_{\mathcal{F}}$$

Moreover, since π is proper-schematic as it is the pull-back of f which is proper-schematic by assumption, the base-change formula holds, so $u^*\pi_* \simeq f_* u'^*$ for all quasi-coherent sheaves, in particular we obtain:

$$\mathbb{T}_g\mathcal{F}_{\mathbf{Map}} \simeq f_* u'^* ev^* \mathbb{T}_{\mathcal{F}}$$

Notice that, in our notation for this proposition, the functor $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(S)$ is denoted by $\Gamma_S(X, -)$, and that the composition $ev \circ u'$ is equivalent to g, so we obtain the desired formula:

$$\mathbb{T}_g\mathcal{F}_{\mathbf{Map}} \simeq \Gamma_S(X, g^*\mathbb{T}_{\mathcal{F}})$$

□

Remark 6.2.6. The same reasoning also applies to describe the tangent of the derived foliation induced by $f_{*,\mathcal{F}ol}$ in theorem 2.2.6. In the setting of this theorem, if $\mathcal{F}$ is a derived foliation on Z relative to S, and $\mathcal{F}_{\mathbf{Res}}$ is its image by $f_{*,\mathcal{F}ol}$, we have:

$$\mathbb{T}_{\mathcal{F},\mathbf{Res}} \simeq \pi_* ev^* \mathbb{T}_{\mathcal{F}}$$

where, in this case, we denote by $ev : f^*f_*Z \to Z$ the co-unit of $f^* \dashv f_*$ itself, and $\pi : f^*f_*Z \to Z$ is the projection. Moreover, for $u : S \to f_*Z$ an S-point of f_*Z, corresponding to an S-linear section $g : X \to Z$ of the structural map $Z \to X$, we have:

$$\mathbb{T}_g\mathcal{F}_{\mathbf{Res}} \simeq \Gamma_S(X, g^*\mathbb{T}_{\mathcal{F}})$$

Note that in this case, the map $g : X \to Z$ is defined in the following commutative diagram:

$$\begin{tikzcd}
& X \arrow[r, "f"] \arrow[ld, "g"'] \arrow[d, "u'"] \arrow[rd, phantom, "\lrcorner", very near start] & S \arrow[d, "u"] \arrow[dd, bend left, equal, "Id_S"] \\
Z \arrow[rd] & f^*f_*Z \arrow[l, "ev"'] \arrow[r, "\pi"] \arrow[d] \arrow[rd, phantom, "\lrcorner", very near start] & f_*Z \arrow[d] \\
& X \arrow[r, "f"'] & S
\end{tikzcd}$$

Remark 6.2.7. While we have provided a description of the tangent and thus also of the cotangent complex of a derived foliation induced by theorem 2.2.1 or theorem 2.2.6, we do not provide a description of the induced mixed differential on the corresponding graded mixed algebra. It should be noted first that the mixed differential comes from the $\mathcal{H}$-action on the perfect linear stack corresponding to the induced derived foliation, so the mixed differential would be computed via this action. However, it should also be noted that the explicit computation of mixed differentials for derived foliations is very difficult, as discussed in [TV25a, remark preceding Def. 2.1.1.3]. This is because the graded mixed algebra of a derived foliation is only quasi-isomorphic to the symmetric algebra of its cotangent. As such, while the mixed differential of the graded mixed algebra itself may have an explicit description, its homotopical transport along the quasi-isomorphism only induces a weaker structure on the symmetric algebra of its cotangent. This structure is not the data of a differential whose square is equal to zero, but the data of a differential equipped with a homotopy from its square to zero, and moreover with infinitely many higher homotopy coherences. The explicit description of this homotopical data is outside the scope of this thesis, but would be required to compute the resulting differential on the derived foliations induced by our theorems. Nonetheless, it is often the case that derived foliations first arise from their tangent bundles rather than their cotangent bundles, in which case the differential on the cotangent bundle corresponds to the dual of the Lie bracket on the tangent of the derived foliation, which moreover often is the restriction of the Lie bracket of the ambient space to this tangent bundle. In those cases, this provides an intuitive description of the mixed differential.

CHAPTER 7

SHIFTED SYMPLECTIC DERIVED FOLIATIONS

7.1 INTRODUCTION

Symplectic geometry emerged from the reformulation of classical mechanics in the nineteenth century [Ham34; Ham35]. In this context, the evolution of a system is described using the *phase space* rather than configuration space: points encode both positions and momenta, and evolution is determined by an energy function and its differential. Its basic mathematical model is the cotangent bundle T^*M of a manifold M, equipped with a canonical 2-form ω, obtained by differentiating the tautological 1-form known as the Liouville form λ, and satisfying three conditions: its *closedness* is the infinitesimal condition $d\omega = 0$ ultimately ensuring energy conservation laws, its *nondegeneracy* is the requirement that the 2-form uniquely identifies vectors by the data of their pairing with other vectors so that Hamiltonian vector fields can be uniquely defined, while its *skew-symmetry* makes it define a Poisson bracket $\{f, g\} = \omega(X_f, X_g)$ with $\{f, g\} = -\{g, f\}$ which ensures that Hamiltonian time evolution is a derivation and conserved quantities arise via $\dot{f} = \{f, H\}$. For a modern exposition of this theory, see [Can01]. These ideas persist beyond cotangent bundles: for example, the passage from a system with symmetries to a reduced phase space is organized by symplectic reduction, where one imposes constraints and then quotients by the symmetry group [MW74; Wei77]. Two difficulties are already visible at this classical level and become systematic in moduli problems. First, constraints can be viewed as intersections, and intersections are frequently non-transverse, so any formalism that discards higher-order tangency data is too rigid for deformation-theoretic questions. Second, quotients by symmetries are typically singular and often retain stabilizers, so a satisfactory geometric model must treat them as stack-like objects rather than smooth manifolds. Derived algebraic geometry addresses both issues by encoding higher automorphisms and computing intersections up to deformation, thereby removing the need for transversality conditions.

This derived perspective leads naturally to *shifted* symplectic geometry, where the symplectic pairing may live in a nonzero cohomological degree. Let X be a derived Artin stack over a characteristic-0 field with perfect cotangent complex $\mathbb{L}_X$, and let $\mathbb{T}_X := \mathbb{L}_X^\vee$ denote the tangent complex. Write $\mathcal{A}^{2,\mathrm{cl}}(X, n)$ for the space of *closed* and *skew-symmetric*

2-forms of degree n on X, defined via the derived de Rham formalism so that "closed" is extra homotopy-coherent structure rather than merely an equation, as detailed in [PTVV]. An element $\omega \in \mathcal{A}^{2,\mathrm{cl}}(X, n)$ is called an *n-shifted symplectic form* if the induced morphism

$$\omega^\flat : \mathbb{T}_X \longrightarrow \mathbb{L}_X[n]$$

is a quasi-isomorphism [PTVV, Def. 1.18], which is the derived analogue of nondegeneracy. In the derived context, this definition is stable under many constructions which provide a supply of examples. The mapping theorem produces systematic degree shifts (an AKSZ-type mechanism): under suitable finiteness and orientation hypotheses, mapping stacks from a d-dimensional oriented source into an n-shifted symplectic target are $(n-d)$-shifted symplectic [PTVV; Cal15]. The derived Lagrangian intersection theorem produces standard geometric examples, such as derived critical loci which carry canonical (-1)-shifted symplectic structures. Other standard geometric examples include moduli stacks of perfect complexes or of G-local systems which acquire shifted symplectic structures determined by Calabi-Yau or orientation data [PTVV]. On the Poisson side, nondegenerate shifted Poisson structures coincide with shifted symplectic structures, and the shifted framework is well suited for deformation-quantization statements in derived settings [CPTVV].

In many situations of interest, especially after reduction by symmetries, the natural bracket is Poisson but degenerate, so one replaces a single global symplectic form by a *symplectic foliation*: the manifold (or stack) decomposes into *symplectic leaves*, i.e. maximal immersed submanifolds on which the Poisson tensor becomes nondegenerate and hence inverts to a symplectic form; this viewpoint is already central in the local structure theory of Poisson manifolds and in canonical examples such as Lie-Poisson spaces, where the leaves are coadjoint orbits [Wei83]. On a symplectic manifold, a complementary and equally classical structure is a *Lagrangian foliation*, whose leaves are maximally isotropic submanifolds; it appears canonically on cotangent bundles via the foliation by fibers T_q^*M, and in Hamiltonian dynamics via the Liouville-Arnold picture, where regular level sets of a completely integrable system form a foliation by Lagrangian tori admitting (local or global) action-angle coordinates [Dui80; Vai89]. These foliation-theoretic structures motivate derived enhancements for two related reasons:

1. In moduli problems and singular quotients one expects leaf spaces and intersections (e.g. of coisotropic or Lagrangian conditions) to be intrinsically derived, so that "leaves" should be treated as derived symplectic substacks rather than smooth manifolds.
2. In *geometric quantization*, the choice of a polarization is precisely a choice of Lagrangian foliation, so a derived theory of Lagrangian/coisotropic structures is a natural prerequisite for formulating quantization procedures that are stable under derived intersections and stacky phenomena [Kos70; Sou70; Woo92; MS18; Saf17].

7.2 DEFINITIONS

In this section, we will define the notions of shifted symplectic derived foliations and Lagrangian derived foliations.

7.2.1 Forms on Derived Foliations

We will start by generalizing from [PTVV, Sec. 1] the notion of n-shifted p-elements of a graded quasi-coherent complex E.

Reminder 7.2.1. Let Z be a derived stack and $E \in \mathrm{QCoh}(Z)$. We recall that $|E| \in \mathcal{S}paces$ denotes the realization of E, defined as the mapping space $\mathbf{Map}_{k-Mod}(k, q_*E)$, where $q : Z \to \mathrm{Spec}(k)$ is the terminal map. Equivalently, q_*E is $\Gamma(E, Z)$ the global sections of E, and by the adjunction $q^* \dashv q_*$ we also have $|E| = \mathbf{Map}_{\mathrm{QCoh}(Z)}(\mathcal{O}_Z, E)$. Note that this association is canonically functorial in E.

Definition 7.2.2. *Let Z be a derived stack and $\mathcal{E} \in \mathrm{QCoh}(Z)^{gr} = \mathrm{QCoh}(Z \times B\mathbb{G}_m)$ (see corollary 5.3.9). For $p \in \mathbb{N}$ and $n \in \mathbb{Z}$, we denote by $\mathcal{E}l^p(\mathcal{E}, n)$ the space of n-shifted p-elements of $\mathcal{E}$, defined as $|\mathcal{E}(p)[n]|$. Note that equivalently we have:*

$$\mathcal{E}l^p(\mathcal{E}, n) \simeq \mathbf{Map}_{\mathrm{QCoh}(Z)^{gr}}(\mathcal{O}_Z(p)[-n], \mathcal{E})$$

where $\mathcal{O}_Z(p)[-n]$ denotes $\mathcal{O}_Z$ viewed as a graded complex in pure weight p and shifted by $-n$ in cohomological degree.

Definition 7.2.3. *Let Z be a derived stack and $E \in \mathrm{QCoh}(Z)$. Let $\mathcal{E} := \mathrm{Sym}_{\mathcal{O}_Z}(E)$ be the free symmetric algebra on E equipped with its canonical grading. In this case, we will denote $\mathcal{A}^p(E^\vee, n) := \mathcal{E}l^p(\mathcal{E}, n)$. Note that by unfolding the definitions, this space is canonically equivalent to $\mathbf{Map}_{\mathrm{QCoh}(Z)}(\mathcal{O}_Z, \mathrm{Sym}^p(E)[n])$.*

Remark 7.2.4. An element $\omega \in \mathcal{A}^p(E^\vee, n)$ corresponds to a map $\mathcal{O}_Z \to \mathrm{Sym}^p(E)[n]$. When E is dualizable, this corresponds to a map $\mathrm{Sym}^p(E^\vee)[n] \to \mathcal{O}_Z$, i.e. an n-shifted p-form on $E^\vee$, hence the notation. Moreover, note that when $p = 2$, this induces in turn to a map $\omega^\flat : E^\vee \to E[n]$.

Definition 7.2.5. *Let Z be a derived stack and $E \in \mathrm{Perf}(Z)$ be dualizable. We will denote by $\mathcal{A}^2(E^\vee, n)^{nd}$ the subspace of $\mathcal{A}^2(E^\vee, n)$ spanned by elements $\omega : \mathcal{O}_Z \to \mathrm{Sym}^2(E)[n]$ such that the associated map $\omega^\flat : E^\vee \to E[n]$ is an equivalence. Such elements are called the non-degenerate n-shifted 2-forms of $E^\vee$.*

Let now $Z \to X$ be a map of derived stacks. As in [PTVV, Def. 1.2], and using the equivalences from corollary 5.3.9, let $\mathcal{E} \in \epsilon - dg^{gr}(Z/X) \simeq \mathrm{QCoh}([\widetilde{\mathcal{L}^{gr}}(Z/X)/\mathcal{H}])$. We can associate to it two graded complexes on Z, functorially in $\mathcal{E}$. Let us start by a remark to see how they are defined.

Remark 7.2.6. Let $Z \to X$ be a map of derived stacks. Recall from definition 4.3.2 that we have a short split exact sequence of abelian group schemes:

$$1 \to B\mathbb{G}_a \to \mathcal{H} \leftrightarrows \mathbb{G}_m \to 1$$

In this remark, we will call $\rho : \mathcal{H} \to \mathbb{G}_m$ and $\iota : \mathbb{G}_m \to \mathcal{H}$ the splitting of this short exact sequence, so $\rho \circ \iota = Id_{\mathbb{G}_m}$. Since they are group morphisms, they admit deloopings. We will call $p = B(\rho) : B\mathcal{H} \to B\mathbb{G}_m$ and $i = B(\iota) : B\mathbb{G}_m \to B\mathcal{H}$ the corresponding deloopings, and by functoriality of B they also verify $p \circ i = Id_{B\mathbb{G}_m}$. Consider the following commutative diagram:

$$\begin{tikzcd}
Z \times B\mathbb{G}_m \arrow[rd, "r"] & \\
{[\widetilde{\mathcal{L}^{gr}}(Z/X)/\mathbb{G}_m]} \arrow[u, "q"] \arrow[r, "h'"'] \arrow[d, shift right, "i'"'] \arrow[rd, phantom, "\lrcorner", very near start] & B\mathbb{G}_m \arrow[d, shift right, "i"'] \\
{[\widetilde{\mathcal{L}^{gr}}(Z/X)/\mathcal{H}]} \arrow[r, "h"'] \arrow[u, shift right, "p'"'] \arrow[ru, phantom, "\urcorner", very near start] & B\mathcal{H} \arrow[u, shift right, "p"']
\end{tikzcd}$$

- Then, under the equivalences $\text{QCoh}(B\mathbb{G}_m) \simeq dg^{gr}$ of remark 4.3.1 and $\text{QCoh}(B\mathcal{H}) \simeq \epsilon - dg^{gr}$ of remark 4.3.3, we have the identification of functors $i^* \simeq -^{gr}$ and $p_* \simeq NC^w$, see [PTVV, Def. 1.2 and Rmk. 1.5] for the definitions in this case. Moreover, since $p \circ i \simeq Id_{B\mathbb{G}_m}$, $i^* \circ p^* \simeq Id_{dg^{gr}}$, and starting from the co-unit $p^*p_* \implies Id_{\epsilon - dg^{gr}}$, the application of i^* induces a natural transformation $p_* \implies i^*$, i.e. a natural transformation $NC^w \implies -^{gr}$.
- Moreover, note that since i is a finite map by the point 2. of lemma 5.2.2, the base-change formula holds in the bottom square of the above diagram formed by i and i'. As such, starting now from $\mathcal{E} \in \epsilon - dg^{gr}(Z/X) \simeq \text{QCoh}([\widetilde{\mathcal{L}^{gr}}(Z/X)/\mathcal{H}])$, we can compute its underlying graded quasi-coherent complex over Z as $q_*i'^*\mathcal{E}$, and then its graded global sections as $r_*q_*i'^*\mathcal{E}$. By commutativity of this diagram and using the base-change formula, this is equivalent to $h'_*i'^*E$ and to $i^*h_*\mathcal{E} = (h_*E)^{gr}$. In the meantime, we can compute its weighted negative cyclic complex over Z as $q_*p'^*E$, and the graded global sections of this complex as $r_*q_*p'^*E$. By commutativity of the diagram, the later graded global sections coincide with $h'_*p'_*E$ and with $p_*h_*E = NC^w(h_*E)$. In this sense, in each case, the different ways one could think of to compute global sections actually coincide, which proves that they generalize the definitions of [PTVV, Def. 1.2], and justifies using the same name for these functors, which will be done in the following definition.

As a side note, from a conceptual point of view, the functors defined above and the canonical natural transform between them come from natural morphisms between the groups $\mathcal{H}$ and $\mathbb{G}_m$ that classify respectively graded mixed and graded structures under Tannaka duality. Moreover, in a sense, the above geometric discussion also implies that these functors commute with global sections. We will not make the statement of this fact more explicit here, as it is not needed in this thesis, however it suggests that a similar behaviour could happen for other structures classified by Tannaka duality.

When considering graded mixed modules, we can use the above remark to define the space of n-shifted p-forms which are closed under the mixed differential.

Definition 7.2.7. *Let $Z \to X$ be a map of derived stacks. As in [PTVV, Def. 1.2], using the equivalences from corollary 5.3.9, let $\mathcal{E} \in \epsilon - dg^{gr}(Z/X) \simeq \mathrm{QCoh}([\widetilde{\mathcal{L}^{gr}}(Z/X)/\mathcal{H}])$. Using the above remark, we associate to it the two following graded complexes, functorially in $\mathcal{E}$:*

1. *We will denote by $\mathcal{E}^{gr} := q_* i'^* \mathcal{E} \in dg^{gr}(Z)$ the underlying graded quasi-coherent sheaf on Z.*
2. *We will denote by $NC^w(\mathcal{E}) := q_* p'_* \mathcal{E} \in dg^{gr}(Z)$ the weighted negative cyclic complex.*
3. *There is a canonical natural transform $NC^w \implies -^{gr}$.*

Definition 7.2.8. *Let $Z \to X$ be a map of derived stacks, and $\mathcal{E} \in \epsilon - dg^{gr}(Z/X)$.*

1. *We will call $\mathcal{E}l^p(\mathcal{E}, n) := \mathcal{E}l^p(\mathcal{E}^{gr}, n)$ the n-shifted p-elements of E.*
2. *We will call $\mathcal{E}l^{p,cl}(\mathcal{E}, n) := \mathcal{E}l^p(NC^w(\mathcal{E}), n)$ the closed n-shifted p-elements of E.*

Unfolding the definitions, we obtain:

$$\mathcal{E}l^p(\mathcal{E}, n) \simeq \mathbf{Map}_{dg^{gr}(Z)}(\mathcal{O}_Z(p)[-n], \mathcal{E}^{gr})$$
$$\mathcal{E}l^{p,cl}(\mathcal{E}, n) \simeq \mathbf{Map}_{\epsilon-dg^{gr}(Z/X)}(\mathcal{O}_Z(p)[-n], \mathcal{E})$$

Remark 7.2.9. In the above situation, the natural transform $NC^w \implies -^{gr}$ described in the previous remark induces a map of spaces $\mathcal{E}l^{p,cl}(\mathcal{E}, n) \to \mathcal{E}l^p(\mathcal{E}, n)$ which is natural in $\mathcal{E}$. This transformation coincides with the application of the forgetful functor $\epsilon - dg^{gr}(Z/X) \to dg^{gr}(Z)$ to the mapping space $\mathbf{Map}_{\epsilon-dg^{gr}(Z/X)}(\mathcal{O}_Z(p)[-n], \mathcal{E})$. If moreover $\mathcal{E}^{gr} \simeq \mathrm{Sym}_{\mathcal{O}_Z}(\mathbb{L}[1])$ for some $\mathbb{L} \in \mathrm{QCoh}(Z)$, then $\mathcal{E}^{gr}(p) \simeq \wedge^p(\mathbb{L})[p]$, so $NC^w(\mathcal{E})(p)[n-p]$ has a canonical map to $\mathcal{E}^{gr}(p)[n-p] \simeq \wedge^p(\mathbb{L})[n]$. Finally if $\mathbb{L}$ is dualizable with dual $\mathbb{T}$, then $\mathcal{E}l^p(\mathcal{E}, n-p) \simeq \mathcal{A}^p(\mathbb{T}, n)$. Also note that while $\mathcal{E}l^p(\mathcal{E}, n-p)$ does not depend on the choice of the mixed differential of $\mathcal{E}$, $\mathcal{E}l^{p,cl}(\mathcal{E}, n-p)$ does depend on it. Morally this is because its elements are closed with respect to the mixed differential of $\mathcal{E}$.

Using the above definitions and remarks, we define the new notion of n-shifted symplectic structures on a derived foliation.

Definition 7.2.10. *Let $Z \to X$ be a map of derived stacks locally of finite presentation, and let $\mathcal{F} \in \mathcal{F}ol(Z/X)$, i.e. $\mathcal{F}^{gr} \simeq \mathrm{Sym}_{\mathcal{O}_Z}(\mathbb{L}_{\mathcal{F}}[1])$, where $\mathbb{L}_{\mathcal{F}}$ is dualizable with dual $\mathbb{T}_{\mathcal{F}}$.*

1. *We denote by $\mathcal{A}^p(\mathcal{F}, n) := \mathcal{E}l^p(\mathcal{F}, n-p)$. By the above remark, it is equivalent to $\mathbf{Map}_{\mathrm{QCoh}(Z)}(\mathcal{O}_Z, \wedge^p(\mathbb{L}_{\mathcal{F}})[n])$.*

2. *When $p = 2$, we set $\mathcal{A}^2(\mathcal{F}, n)^{nd}$ the subspace of maps $\mathcal{O}_Z \to \wedge^2(\mathbb{L}_\mathcal{F})[n]$ such that the associated map $\mathbb{T}_\mathcal{F} \to \mathbb{L}_\mathcal{F}[n]$ is an equivalence. This is compatible with definition 7.2.5.*
3. *We set $\mathcal{A}^{p,cl}(\mathcal{F}, n) := \mathcal{E}l^{p,cl}(\mathcal{F}, n-p) \simeq \mathbf{Map}_{\epsilon-dg^{gr}(Z/X)}(\mathcal{O}_Z(p)[-n], \mathcal{F})$. It comes equipped with a canonical map $\mathcal{A}^{p,cl}(\mathcal{F}, n) \to \mathcal{A}^p(\mathcal{F}, n)$ as in the above remark.*
4. *We denote by $\mathbb{S}ymp(\mathcal{F}, n)$ the following pull-back space:*

$$\begin{tikzcd}
\mathbb{S}ymp(\mathcal{F}, n) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A}^{2,cl}(\mathcal{F}, n) \arrow[d] \\
\mathcal{A}^2(\mathcal{F}, n)^{nd} \arrow[r, hook] & \mathcal{A}^2(\mathcal{F}, n)
\end{tikzcd}$$

The data of a derived foliation $\mathcal{F}$ equipped with an n-shifted symplectic form $\omega \in \mathbb{S}ymp(\mathcal{F}, n)$ will be called an n-shifted symplectic derived foliation.

Remark 7.2.11. As in [PTVV, Rmk. 1.19], since $\mathbb{L}_\mathcal{F}$ is perfect it is bounded in amplitude and thus, assuming $\mathbb{L}_\mathcal{F}$ is not 0, there can be at most one $n \in \mathbb{Z}$ for which $\mathbb{S}ymp(\mathcal{F}, n)$ is non-empty, otherwise it would be periodic and thus unbounded. More precisely, if $\mathbb{L}_\mathcal{F}$ has amplitude $[-a, b]$ with $a, b \geq 0$, then at most $\mathbb{S}ymp(\mathcal{F}, b-a)$ can be non-empty. Note that this condition is independent of the choice of the mixed differential ϵ. Moreover, when $\mathcal{F} = \mathbf{DR}(Z)$, all the above definitions coincide with those of [PTVV]. Finally, note that the definition of a w-locked d-shifted symplectic fibration $f : Z \to B$ as in [Par24, Sec. 2.2 (D1)] induces a d-shifted symplectic structure on the integrable derived foliation $\mathbf{DR}(Z/B)$ as in the above definition. In this sense, shifted symplectic fibrations can be thought of as exactly the integrable shifted symplectic derived foliations.

We can construct derived stacks $\mathfrak{A}^p_Z[n]$ and $\mathfrak{A}^{p,cl}_{Z/X}[n]$ over $Z \times B\mathbb{G}_m$ and $[\widetilde{\mathcal{L}^{gr}}(Z/X)/\mathcal{H}]$ respectively that represent the functors $\mathcal{A}^p(-, n)$ and $\mathcal{A}^{p,cl}(-, n)$.

Definition 7.2.12. *Let $Z \to X$ be a map of derived stacks which is locally of finite presentation, consider the following derived stacks:*

1. *Let* $\mathfrak{A}^p_Z[n] := \mathrm{Spec}_{Z \times B\mathbb{G}_m} \mathrm{Sym}_{\mathcal{O}_{Z \times B\mathbb{G}_m}} \mathcal{O}_Z(p)[p-n]$
2. *Let* $\mathfrak{A}^{p,cl}_{Z/X}[n] := \mathrm{Spec}_{[\widetilde{\mathcal{L}^{gr}}(Z/X)/\mathcal{H}]} \mathrm{Sym}_{\mathcal{O}_{[\widetilde{\mathcal{L}^{gr}}(Z/X)/\mathcal{H}]}} \mathcal{O}_Z(p)[p-n]$

Hence, by definition, given $\mathcal{F} \in \mathcal{F}ol(Z/X)$ with associated perfect linear stack $\mathbb{V}(\mathcal{F})$ in $\mathrm{PerfLinSt}_Z$ *equipped with its canonical $\mathcal{H}$-action and induced $\mathbb{G}_m$-action, we have canonical equivalences:*

$$\mathcal{A}^p(\mathcal{F}, n) \simeq \mathbf{Map}_{Z \times B\mathbb{G}_m}([\mathbb{V}(\mathcal{F})/\mathbb{G}_m], \mathfrak{A}^p_Z[n])$$
$$\mathcal{A}^{p,cl}(\mathcal{F}, n) \simeq \mathbf{Map}_{[\widetilde{\mathcal{L}^{gr}}(Z/X)/\mathcal{H}]}([\mathbb{V}(\mathcal{F})/\mathcal{H}], \mathfrak{A}^{p,cl}_{Z/X}[n])$$

Definition 7.2.13. *In the above situation, we define the $\mathcal{H}$-equivariant derived stack $\mathbf{A}^p_{Z/X}[n]$ over $\widetilde{\mathcal{L}^{gr}}(Z/X)$ by the following pull-back diagram:*

$$\begin{tikzcd}
\mathbf{A}^p_{Z/X}[n] \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathfrak{A}^p_Z[n] \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathfrak{A}^{p,cl}_{Z/X}[n] \arrow[d] \\
\widetilde{\mathcal{L}^{gr}}(Z/X) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & {[\widetilde{\mathcal{L}^{gr}}(Z/X)/\mathbb{G}_m]} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & {[\widetilde{\mathcal{L}^{gr}}(Z/X)/\mathcal{H}]} \arrow[dd] \\
Z \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Z \times B\mathbb{G}_m \arrow[d] & \\
\bullet \arrow[r] & B\mathbb{G}_m \arrow[r] & B\mathcal{H}
\end{tikzcd}$$

In which the square in the top right corner is a pull-back because relative spectra are stable by base-change by reminder 5.1.4.

In particular, since its quotients by $\mathbb{G}_m$ and $\mathcal{H}$ recover the previous stacks as shown in this diagram, then for $\mathbb{V}(\mathcal{F}) \in \mathcal{F}ol(Z/X)$ a derived foliation viewed as an $\mathcal{H}$-equivariant perfect linear stack on Z, we have the following canonical equivalences:

$$\mathcal{A}^p(\mathcal{F}, n) \simeq \mathbf{Map}_{\mathrm{dSt}\,/Z-G_m}(\mathbb{V}(\mathcal{F}), \mathbf{A}^p_{Z/X})$$
$$\mathcal{A}^{p,cl}(\mathcal{F}, n) \simeq \mathbf{Map}_{\mathrm{dSt}\,/\widetilde{\mathcal{L}^{gr}}(Z/X)-\mathcal{H}}(\mathbb{V}(\mathcal{F}), \mathbf{A}^p_{Z/X})$$

and, under those equivalences, the canonical map $\mathcal{A}^{p,cl}(\mathcal{F}, n) \to \mathcal{A}^p(\mathcal{F}, n)$ is obtained by application of the forgetful functor $-^{Z-\mathbb{G}_m} : \mathrm{dSt}\,/\widetilde{\mathcal{L}^{gr}}(Z/X) - \mathcal{H} \to \mathrm{dSt}\,/Z - \mathbb{G}_m$. In this sense, p-forms are $\mathbb{G}_m$-equivariant functions, while closed p-forms are $\mathcal{H}$-equivariant functions.

Remark 7.2.14. In this remark, for a derived stack S, we will denote by $\mathbb{V}_S = \mathrm{Spec}_S \circ \mathrm{Sym}_{\mathcal{O}_S} : \mathrm{QCoh}(S)^{\mathrm{op}} \to \mathrm{dSt}\,/S$ without the canonical $\mathbb{G}_m$-action.

In the above situation, since $\mathfrak{A}^p_Z[n] \to Z \times B\mathbb{G}_m$ is relatively affine by definition, in fact it is $\mathbb{V}_{Z\times B\mathbb{G}_m}(\mathcal{O}_Z(p)[p-n])$ as derived stacks over $Z \times B\mathbb{G}_m$, by pull-back $\mathbf{A}^p_{Z/X}[n] \to Z$ is also relatively affine (see reminder 5.1.4), and in fact it is $\mathbb{V}_Z(\mathcal{O}_Z(p)[p-n])$ as derived stacks over Z, where $\mathcal{O}_Z(p)[p-n]$ is now viewed in $\mathrm{QCoh}(Z)$ by forgetting its grading, i.e. by pull-back along $Z \to Z \times B\mathbb{G}_m$. Note that, after forgetting the grading, this module is simply equivalent to $\mathcal{O}_Z[p-n]$ in $\mathrm{QCoh}(Z)$. Moreover, note that $\mathcal{O}_Z(p)[p-n] \in \mathrm{QCoh}(Z \times B\mathbb{G}_m)$ is dualizable as a graded module since it is concentrated in the single weight p and the module $\mathcal{O}_Z[p-n] \in \mathrm{QCoh}(Z)$ in this weight is dualizable in $\mathrm{QCoh}(Z)$. See the proof of [Pav23, Prop. 1.1.8] for the fact that this condition implies dualizability in $\mathrm{QCoh}(Z \times B\mathbb{G}_m)$. Note that more generally, their proof implies that an object $E = (E(n))_{n\in\mathbb{Z}} \in \mathrm{QCoh}(Z \times B\mathbb{G}_m) \simeq dg^{gr}(Z)$ is dualizable if and only if $E(n) = 0$ outside a finite set of weights, or equivalently E is supported on finitely many weights, and

$E(n)$ is dualizable in each weight. As such, the induced module $\mathcal{O}_Z(p)[n-p] \in \mathrm{QCoh}(Z)$ is also dualizable by pull-back, which could also be seen from the fact that it is equivalent to $\mathcal{O}_Z[p-n]$ in this category.

Therefore, $\mathbf{A}^p_{Z/X}[n] \simeq \mathbb{V}_Z(\mathcal{O}_Z[p-n])$ as derived stacks over Z, $\mathcal{O}_Z[p-n] \in \mathrm{Perf}(Z)$, it has canonical $\mathcal{H}$- and $\mathbb{G}_m$-actions, and a canonical $\mathcal{H}$-equivariant map to $\widetilde{\mathcal{L}^{gr}}(Z/X)$. In fact, as derived stacks over Z, it is standard that $\mathrm{Spec}_Z(\mathrm{Sym}_{\mathcal{O}_Z}(\mathcal{O}_Z[p-n])) =: \mathbb{A}^1_Z[n-p]$ the $(n-p)$-shifted affine line over Z, and recovers the usual affine line when $n-p=0$. In conclusion, as derived sacks over Z, one has:

$$\mathbf{A}^p_{Z/X}[n] \simeq \mathbb{A}^1_Z[n-p] \simeq \mathbb{V}_Z(\mathcal{O}_Z[p-n])$$

However, it is very important to note that this is not a perfect linear stack in the sense of belonging to $\mathrm{PerfLinSt}_Z$ as in definition 5.1.10, and thus it does not define a derived foliation on Z relative to X. This is because, even though this derived stack does have canonical $\mathcal{H}$- and $\mathbb{G}_m$-actions, by construction they are discting from the canonical grading of $\mathbb{V}_Z(\mathcal{O}_Z[p-n])$. This distinct grading has been designed so that the equivalences in the above definition hold and thus $\mathbf{A}^p_{Z/X}[n]$ can represent $\mathcal{A}^p(-,n)$ and $\mathcal{A}^{p,cl}(-,n)$, which would not be the case if the canonical grading had been used instead.

Remark 7.2.15. Let $Z \to X$ be locally of finite presentation, and assume that it admits a relative cotangent $\mathbb{L}_{Z/X} \in \mathrm{QCoh}(Z)$. Then, as explained in corollary 5.3.19, we can construct $\mathrm{Sym}_{\mathcal{O}_Z}(\mathbb{L}_{Z/X}[1]) = \mathbf{DR}(Z/X) \in \epsilon - cdga^{gr}(Z/X) = \mathrm{CAlg}(\mathrm{QCoh}([\widetilde{\mathcal{L}^{gr}}(Z/X)/\mathcal{H}]))$ the relative de Rham algebra equipped with its canonical graded mixed structure. Moreover, by the same result, this defines the terminal derived foliation on Z relative to X, in the sense that $\widetilde{\mathcal{L}^{gr}}(Z/X) \simeq \mathrm{Spec}_Z(\mathbf{DR}(Z/X))$ as $\mathbb{G}_m$-equivariant derived stacks over Z. This allows us to generalize the definitions of [PTVV] to the relative context.

Definition 7.2.16. *Let $Z \to X$ be locally of finite presentation and assume that it admits a relative cotangent $\mathbb{L}_{Z/X}$ which is thus perfect. We apply definition 7.2.10 to the derived foliation $\mathbf{DR}(Z/X)$ as above to define the following forms on Z relative to X:*

1. $\mathcal{A}^p(Z/X,n) := \mathbf{Map}_{\mathrm{dSt}/Z-\mathbb{G}_m}(\widetilde{\mathcal{L}^{gr}}(Z/X), \mathbf{A}^p_{Z/X}[n])$. *Note that this space is canonically equivalent to $\mathbf{Map}_{\mathrm{QCoh}(Z)}(\mathcal{O}_Z, \wedge^p\mathbb{L}_{Z/X}[n])$ after unfolding the definitions.*
2. *Since $\mathbb{L}_{Z/X}$ is perfect, we can define $\mathcal{A}^2(Z/X,n)^{nd}$ to be the subspace of $A^2(Z/X,n)$ spanned by maps $\mathcal{O}_Z \to \wedge^2\mathbb{L}_{Z/X}[n]$ whose associated map $\mathbb{T}_{Z/X} \to \mathbb{L}_{Z/X}[n]$ is an equivalence.*
3. $\mathcal{A}^{p,cl}(Z/X,n) := \mathbf{Map}_{\mathrm{dSt}/\widetilde{\mathcal{L}^{gr}}(Z/X)-\mathcal{H}}(\widetilde{\mathcal{L}^{gr}}(Z/X), \mathbf{A}^p_{Z/X}[n])$. *Note that this space is canonically equivalent to $\mathbf{Map}_{\epsilon-dg^{gr}(Z/X)}(\mathcal{O}_Z(p)[-n], \mathbf{DR}(Z/X))$ after unfolding the definitions, and it comes equipped with a canonical map $\mathcal{A}^{p,cl}(Z/X,n) \to \mathcal{A}^p(Z/X,n)$.*
4. $\mathbb{S}ymp(Z/X,n) := \mathcal{A}^{2,cl}(Z/X,n) \times_{\mathcal{A}^2(Z/X,n)} \mathcal{A}^2(Z/X,n)^{nd}$

7.2.2 Lagrangian Derived Foliations

We will now turn to the complementary direction of Lagrangian derived foliations. Intuitively, they are derived foliations $\mathcal{F}$ on an n-shifted symplectic derived stack Z such that its leaves are equipped with Lagrangian structures with respect to the symplectic form, or equivalently such that the formal map to the formal leaf space $Z \to [Z/\mathcal{F}]$ is equipped with the structure of a Lagrangian fibration. However, we will give a definition that does not involve formal geometry, while restricting ourselves to the case where $Z \to X$ admits a relative cotangent as above so that differential forms on Z relative to X are defined.

Definition 7.2.17. *Let $Z \to X$ be locally of finite presentation and assume that it admits a relative cotangent $\mathbb{L}_{Z/X}$. Let $\mathcal{F} \in \mathcal{F}ol(Z/X)$. We define maps $-_{|\mathcal{F}} : \mathcal{A}^{p,cl}(Z/X, n) \to \mathcal{A}^{p,cl}(\mathcal{F}, n)$ and $-_{|\mathcal{F}} : \mathcal{A}^{p}(Z/X, n) \to \mathcal{A}^{p}(\mathcal{F}, n)$ by the following commutative diagram:*

$$\begin{tikzcd}
\mathcal{A}^{p,cl}(Z/X,n) \arrow[r, "-_{|\mathcal{F}}"] \arrow[d, dash, "\simeq"'] & \mathcal{A}^{p,cl}(\mathcal{F},n) \arrow[d, dash, "\simeq"] \\
\mathbf{Map}_{\mathrm{dSt}\,/\widetilde{\mathcal{L}^{gr}}(Z/X)-\mathcal{H}}(\widetilde{\mathcal{L}^{gr}}(Z/X), \mathbf{A}^p[n]) \arrow[r] \arrow[d] & \mathbf{Map}_{\mathrm{dSt}\,/\widetilde{\mathcal{L}^{gr}}(Z/X)-\mathcal{H}}(\mathbb{V}(\mathcal{F}), \mathbf{A}^p[n]) \arrow[d] \\
\mathbf{Map}_{\mathrm{dSt}\,/Z-\mathbb{G}_m}(\widetilde{\mathcal{L}^{gr}}(Z/X), \mathbf{A}^p[n]) \arrow[r] \arrow[d, dash, "\simeq"'] & \mathbf{Map}_{\mathrm{dSt}\,/Z-\mathbb{G}_m}(\mathbb{V}(\mathcal{F}), \mathbf{A}^p[n]) \arrow[d, dash, "\simeq"] \\
\mathcal{A}^{p}(Z/X,n) \arrow[r, "-_{|\mathcal{F}}"] & \mathcal{A}^{p}(\mathcal{F},n)
\end{tikzcd}$$

where the maps in the two middle rows are given by pre-composition with the structural $\mathcal{H}$-equivariant (and thus $\mathbb{G}_m$-equivariant) map $\mathbb{V}(\mathcal{F}) \to \widetilde{\mathcal{L}^{gr}}(Z/X)$ coming from the anchor map $a_{\mathcal{F}} : \mathbb{L}_{Z/X} \to \mathbb{L}_{\mathcal{F}}$. Alternatively, note that this map is also the terminal map from $\mathbb{V}(\mathcal{F})$ to the terminal derived foliation $\widetilde{\mathcal{L}^{gr}}(Z/X)$. Both maps are called the restriction of forms to $\mathcal{F}$*, and the commutativity of the diagram ensures that no confusion can arise by using the same name, as they compute the same underlying forms whether we view the initial forms as closed or not.*

Remark 7.2.18. As noted above, the structural map $\mathbb{V}(\mathcal{F}) \to \widetilde{\mathcal{L}^{gr}}(Z/X)$ comes from the anchor map $a_{\mathcal{F}} : \mathbb{L}_{Z/X} \to \mathbb{L}_{\mathcal{F}}$. As such, we can give a more explicit description of $-_{|\mathcal{F}}$: the map $a_{\mathcal{F}}$ induces by functoriality maps $f : \wedge^p(\mathbb{L}_{Z/X}) \to \wedge^p(\mathbb{L}_{\mathcal{F}})$ and $\phi : NC^w(\mathbb{L}_{Z/X})(p)[n-p] \to NC^w(\mathbb{L}_{\mathcal{F}})(p)[n-p]$, and given ω a (closed) form viewed as a map $\mathcal{O}_Z \to \wedge^p(\mathbb{L}_{Z/X})$ (resp. $\mathcal{O}_Z \to NC^w(\mathbb{L}_{Z/X})(p)[n-p]$), we obtain that $\omega_{|\mathcal{F}} = f \circ \omega$ (resp. $\phi \circ \omega$). Moreover, the underlying form of the closed form $\phi \circ \omega$ is exactly $f \circ \omega$ by commutativity of the diagram above.

Definition 7.2.19. *Let $Z \to X$ be locally of finite presentation, admitting a relative cotangent $\mathbb{L}_{Z/X}$, and equipped with an n-shifted symplectic form $\omega \in \mathbb{S}ymp(Z/X, n)$. Let $\mathcal{F} \in \mathcal{F}ol(Z/X)$. An* isotropic structure *on $\mathcal{F}$ is the data of a homotopy to zero of $\omega_{|\mathcal{F}}$ in $\mathcal{A}^{2,cl}(\mathcal{F}, n)$, i.e. we set the space:*

$$\mathrm{Isot}(\mathcal{F}, \omega) := Path_{0,\omega_{|\mathcal{F}}}(\mathcal{A}^{2,cl}(\mathcal{F}, n))$$

Remark 7.2.20. In the above situation, let $h \in \mathrm{Isot}(\mathcal{F}, \omega)$. Note that by remark 7.2.18 applied when $p = 2$, we obtain that ω and $\omega_{|\mathcal{F}}$ are related by the following commutative diagram:

$$\begin{tikzcd}
\mathbb{T}_{\mathcal{F}} \arrow[r, "\omega_{|\mathcal{F}}"] \arrow[d, "a^{\vee}"'] & \mathbb{L}_{\mathcal{F}}[n] \\
\mathbb{T}_{Z/X} \arrow[r, "\omega"'] & \mathbb{L}_{Z/X}[n] \arrow[u, "a[n]"']
\end{tikzcd}$$

where $a : \mathbb{L}_{Z/X} \to \mathbb{L}_{\mathcal{F}}$ is the anchor map. In particular, since $\omega_{|\mathcal{F}} \simeq 0$ by h, we observe that the map $\omega \circ a^{\vee} : \mathbb{T}_{\mathcal{F}} \to \mathbb{L}_{Z/X}[n]$ factors through the fiber of $a[n] : \mathbb{L}_{Z/X}[n] \to \mathbb{L}_{\mathcal{F}}[n]$, i.e. it induces a map $\Theta_h : \mathbb{T}_{\mathcal{F}} \to \mathcal{N}^*_{\mathcal{F}}[n]$, where $\mathcal{N}^*_{\mathcal{F}}$ is the conormal of $\mathcal{F}$, defined as $\mathrm{fib}(a : \mathbb{L}_{Z/X} \to \mathbb{L}_{\mathcal{F}})$, so that it comes equipped with a canonical map $\iota_{\mathcal{F}} : \mathcal{N}^*_{\mathcal{F}} \to \mathbb{L}_{Z/X}$.

Definition 7.2.21. *Let $Z \to X$ be locally of finite presentation, admitting a relative cotangent $\mathbb{L}_{Z/X}$, and equipped with an n-shifted symplectic form $\omega \in \mathbb{S}ymp(Z/X, n)$. Let $\mathcal{F} \in \mathcal{F}ol(Z/X)$. An isotropic structure $h \in \mathrm{Isot}(\mathcal{F}, \omega)$ is a* Lagrangian structure *if the induced map:*

$$\Theta_h : \mathbb{T}_{\mathcal{F}} \to \mathcal{N}^*_{\mathcal{F}}[n]$$

is an equivalence. A derived foliation $\mathcal{F}$ equipped with a Lagrangian structure will be called a Lagrangian derived foliation.

Remark 7.2.22. When the derived foliation $\mathcal{F}$ is integrable and comes from a map $f : Z \to B$ over X, so that in this case $\mathcal{F} = \mathbf{DR}(Z/B)$, the definition of an isotropic (resp. Lagrangian) structure on $\mathcal{F}$ as above coincides with the definition of an isotropic (resp. Lagrangian) fibration structure on f in the terminology of [Saf23] and [Cal19]. In this sense, by definition, Lagrangian fibrations are exactly the integrable Lagrangian derived foliations.

7.3 EXISTENCE THEOREMS

In this section, we will prove existence theorems for shifted symplectic derived foliations and Lagrangian derived foliations.

7.3.1 Intersection of Lagrangian Derived Foliations

Definition 7.3.1. *Let $Z \to X$ be locally of finite presentation and admitting a relative cotangent complex $\mathbb{L}_{Z/X}$, which is thus perfect by local finite presentation. Let $\mathcal{F}, \mathcal{F}' \in \mathcal{F}ol(Z/X)$ be two derived foliations on Z relative to X. Then the derived tensor product of their underlying graded mixed algebras, $\mathcal{F} \otimes_{\mathbf{DR}(Z/X)} \mathcal{F}'$, defines a derived foliation with perfect cotangent $\mathbb{L}_{\mathcal{F}} \oplus_{\mathbb{L}_{Z/X}} \mathbb{L}_{\mathcal{F}'}$ and tangent $\mathbb{T}_{\mathcal{F}} \times_{\mathbb{T}_{Z/X}} \mathbb{T}_{\mathcal{F}'}$. It is called the* intersection derived foliation *of $\mathcal{F}$ and $\mathcal{F}'$.*

Remark 7.3.2. In the above context, the intersection derived foliation of $\mathcal{F}$ and $\mathcal{F}'$ can also be computed from when viewing them as perfect linear stacks. On this side, starting from $\mathbb{V}(\mathcal{F})$ and $\mathbb{V}(\mathcal{F}')$ the associated perfect linear stacks equipped with their structural maps over $\widetilde{\mathcal{L}^{gr}}(Z/X)$, which is the terminal derived foliation in this case, the above construction coincides with their derived intersection $\mathbb{V}(\mathcal{F}) \times_{\widetilde{\mathcal{L}^{gr}}(Z/X)} \mathbb{V}(\mathcal{F}')$.

Notation 7.3.3. In the above context, the intersection derived foliation is denoted as $\mathcal{F} \times_{Z/X} \mathcal{F}'$, or more simply as $\mathcal{F} \times \mathcal{F}'$ when the context is clear.

Theorem 7.3.4 (Theorem 2.3.1)**.** *Let $Z \to X$ be a map of derived stacks which is locally of finite presentation and admits a relative cotangent $\mathbb{L}_{Z/X}$, $\omega \in \mathbb{S}ymp(Z/X, n)$ an n-shifted symplectic form, and $\mathcal{F}, \mathcal{F}' \in \mathcal{F}ol(Z/X)$ be two derived foliations equipped with Lagrangian structures h and h'. Then the intersection derived foliation $\mathcal{F}'' := \mathcal{F} \times_{Z/X} \mathcal{F}' \in \mathcal{F}ol(Z/X)$ inherits an $(n-1)$-shifted symplectic form.*

Proof. Let $a : \mathbb{L}_{Z/X} \to \mathbb{L}_{\mathcal{F}}$ and $a' : \mathbb{L}_{Z/X} \to \mathbb{L}_{\mathcal{F}'}$ be the anchor maps, with duals $a^{\vee} : \mathbb{T}_{\mathcal{F}} \to \mathbb{T}_{Z/X}$ and $a'^{\vee} : \mathbb{T}_{\mathcal{F}'} \to \mathbb{T}_{Z/X}$. By definition of the intersection derived foliation, there are equivalences:

$$\mathbb{L}_{\mathcal{F}''} \simeq \mathbb{L}_{\mathcal{F}} \oplus_{\mathbb{L}_{Z/X}} \mathbb{L}_{F'}$$
$$\mathbb{T}_{F''} \simeq \mathbb{T}_{\mathcal{F}} \times_{\mathbb{T}_{Z/X}} \mathbb{T}_{\mathcal{F}'}$$

We will call $p : \mathbb{T}_{\mathcal{F}''} \to \mathbb{T}_{\mathcal{F}}$ and $p' : \mathbb{T}_{\mathcal{F}''} \to \mathbb{T}_{\mathcal{F}'}$ the two induced projections. Now let $\mathcal{N}^*_{\mathcal{F}} := \mathrm{fib}(\mathbb{L}_{Z/X} \to \mathbb{L}_{\mathcal{F}})$ and $\mathcal{N}^*_{\mathcal{F}'} := \mathrm{fib}(\mathbb{L}_{Z/X} \to \mathbb{L}_{\mathcal{F}'})$ be the conormal complexes of $\mathcal{F}$ and $\mathcal{F}'$, equipped with their canonical maps $\iota_{\mathcal{F}} : \mathcal{N}^*_{\mathcal{F}} \to \mathbb{L}_{Z/X}$ and $\iota_{\mathcal{F}'} : \mathcal{N}^*_{\mathcal{F}'} \to \mathbb{L}_{Z/X}$. By hypothesis, the Lagrangian structures h, h' induce equivalences:

$$\Theta_h : \mathbb{T}_{\mathcal{F}} \xrightarrow{\sim} \mathcal{N}^*_{\mathcal{F}}[n]$$
$$\Theta_{h'} : \mathbb{T}_{\mathcal{F}'} \xrightarrow{\sim} \mathcal{N}^*_{\mathcal{F}'}[n]$$

We can thus form the following diagram, which is commutative by the definitions of $\mathbb{T}_{\mathcal{F}''}$, Θ_h and $\Theta_{h'}$:

$$\begin{tikzcd}
 & & \mathbb{T}_{\mathcal{F}''} \arrow[rr, "p"] \arrow[lld, dashed, "\simeq"'] \arrow[dd, "p'"] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathbb{T}_{\mathcal{F}} \arrow[ld, "\Theta_h\simeq"] \arrow[dd, "a^\vee"] \\
\mathcal{N}^*_{\mathcal{F}'}[n] \times_{\mathbb{L}_{Z/X}[n]} \mathcal{N}^*_{\mathcal{F}}[n] \arrow[rrr] \arrow[dd] \arrow[drrr, phantom, "\lrcorner", very near start] & & & \mathcal{N}^*_{\mathcal{F}}[n] \arrow[dd, "\iota_{\mathcal{F}}[n]"] & \\
 & & \mathbb{T}_{\mathcal{F}'} \arrow[rr, "a'^\vee"] \arrow[lld, "\Theta_{h'}\simeq"'] & & \mathbb{T}_{Z/X} \arrow[ld, "\omega\simeq"] \\
\mathcal{N}^*_{\mathcal{F}'}[n] \arrow[rrr, "\iota_{\mathcal{F}'}[n]"'] & & & \mathbb{L}_{Z/X}[n] &
\end{tikzcd}$$

where the dotted map $\mathbb{T}_{\mathcal{F}''} \to \mathcal{N}^*_{\mathcal{F}}[n] \times_{\mathbb{L}_{Z/X}[n]} \mathcal{N}^*_{\mathcal{F}'}[n]$, which arises by definition of the pull-back, is an equivalence since Θ_h, $\Theta_{h'}$, and ω are equivalences. We can now pass to the following diagram:

$$\begin{tikzcd}
\mathbb{T}_{\mathcal{F}''} \arrow[dr, "\simeq"] & & & \\
 & \mathcal{N}^*_{\mathcal{F}'}[n] \times_{\mathbb{L}_{Z/X}[n]} \mathcal{N}^*_{\mathcal{F}}[n] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{N}^*_{\mathcal{F}}[n] \arrow[r] \arrow[d, "\iota_{\mathcal{F}}[n]"'] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
 & \mathcal{N}^*_{\mathcal{F}'}[n] \arrow[r, "\iota_{\mathcal{F}'}[n]"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{L}_{Z/X}[n] \arrow[r, "a'[n]"] \arrow[d, "a[n]"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "\ulcorner", very near end] & \mathbb{L}_{\mathcal{F}'}[n] \arrow[d] \\
 & 0 \arrow[r] & \mathbb{L}_{\mathcal{F}}[n] \arrow[r] & \mathbb{L}_{\mathcal{F}''}[n]
\end{tikzcd}$$

where the bottom-right square is a push-out since $\mathbb{L}_{\mathcal{F}''} \simeq \mathbb{L}_{\mathcal{F}} \oplus_{\mathbb{L}_{Z/X}} \mathbb{L}_{\mathcal{F}'}$ by definition of the intersection derived foliation and thus this square is also a pull-back since $\mathrm{QCoh}(Z)$ is a stable ∞-category, and the other squares are pull-back by definition of the objects at their corner. By pasting, the outer square is thus also a pull-back, so that we can also identify $\mathbb{L}_{\mathcal{F}''}[n-1] \simeq \mathcal{N}^*_{\mathcal{F}'}[n] \times_{\mathbb{L}_{Z/X}[n]} \mathcal{N}^*_{\mathcal{F}}[n]$, and finally by composition we obtain an equivalence $\omega_{\mathcal{F}''} : \mathbb{T}_{\mathcal{F}''} \to \mathbb{L}_{\mathcal{F}''}[n-1]$ which is our desired $(n-1)$-shifted symplectic form on $\mathcal{F}''$. ☐

7.3.2 Pushforward of Shifted Symplectic Derived Foliations

Let us start by defining and studying $\mathcal{O}$-compact maps, as they are the main ingredient in the proof of the final theorem.

Definition 7.3.5. *Let $f : X \to Y$ be a map of derived stacks. We say that f is* $\mathcal{O}$-compact*, or that X* is relatively $\mathcal{O}$-compact over Y *when f is clear in the context, if the following condition holds:*

For any map $\mathrm{Spec}(A) \to Y$, the pull-back $f_A : X_A := X \times_Y \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ is strictly $\mathcal{O}$-compact over $\mathrm{Spec}(A)$ in the sense of [PTVV, Def. 2.1], i.e. $\mathcal{O}_{X_A}$ is compact in $\mathrm{QCoh}(X_A)$ and $f_{A,*} : \mathrm{QCoh}(X_A) \to A - Mod$ preserves perfect objects.

Remark 7.3.6. First, note that, contrary to [PTVV, Rmk. 2.2], it is not immediate that $\mathcal{O}_X$ is a compact object of $\mathrm{QCoh}(X)$ in general, and we will not investigate this fact as we will not need it here. However, when Y is affine, it is immediate to check that this condition does imply the compactness of $\mathcal{O}_X$, and moreover when $Y = \mathrm{Spec}(k)$ the above definition does recover exactly the definition of $\mathcal{O}$-compactness of [PTVV, Def. 2.1].

Second, consider the push-forward functor $f_*^{\mathrm{Mod}} : \mathcal{O}_X - \mathrm{Mod} \to \mathcal{O}_Y - \mathrm{Mod}$, as in definition 3.4.6. In general, this functor does not preserve quasi-coherent sheaves, but instead if $(-)^{\mathrm{QCoh}} : \mathcal{O}_Y - \mathrm{Mod} \to \mathrm{QCoh}(Y)$ is the quasi-coherator, then the push-forward of quasi-coherent sheaves is given by $f_*^{\mathrm{QCoh}} = (-)^{\mathrm{QCoh}} \circ f_*^{\mathrm{Mod}} : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$, as in proposition 3.4.10. However, we will show in proposition 7.3.9 that the above condition implies that $f_*^{\mathrm{Mod}} : \mathcal{O}_X - \mathrm{Mod} \to \mathcal{O}_Y - \mathrm{Mod}$ does preserve quasi-coherent sheaves in this case, so we need not apply $(-)^{\mathrm{QCoh}}$. This makes the push-forward of quasi-coherent sheaves much better behaved under these conditions, and is a strong conceptual justification behind introducing it. In fact, we will not need to know that $f_{A,*}$ preserve perfect complexes to show this fact, but this extra condition will be necessary for future constructions in this section.

We will now show important properties of $\mathcal{O}$-compactness.

Proposition 7.3.7 (Stability under pull-back)**.** *Let $f : X \to Y$ be an $\mathcal{O}$-compact map of derived stacks, and $q : V \to Y$ be any map of derived stacks, then the pull-back $f' = q^*(f) : q^*X \to V$ is $\mathcal{O}$-compact.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
q^*X \arrow[r, "f'"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, "q"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Let $\mathrm{Spec}(A) \to V$ be any map from a derived affine to V, then consider the pull-back of this derived affine over f':

$$\begin{tikzcd}
X_A \arrow[r, "f'_A"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spec}(A) \arrow[d] \\
q^*X \arrow[r, "f'"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, "q"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

by pasting, the outer square is a pull-back, so $f'_A : X_A \to \mathrm{Spec}(A)$ is exactly the map f_A obtained in definition 7.3.5, so it is strictly $\mathcal{O}$-compact by assumption on f, and thus f' is $\mathcal{O}$-compact too. ☐

Lemma 7.3.8 (Base-change in an affine case)**.** *Let $f : X \to \operatorname{Spec}(A)$ be an $\mathcal{O}$-compact map of derived stacks over an affine, and $q : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ be an arbitrary map of affines. Let $X_B = q^*X = X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(B)$, $f' : X_B \to \operatorname{Spec}(B)$ be the pull-back of f, and $p : X_B \to X$ be the pull-back of q. Then, for any $E \in \operatorname{QCoh}(X)$, the canonical base-change map $q^*f_*E \to f'_*p^*E \in B - \operatorname{Mod}$ is an equivalence, i.e. the base-change formula holds.*

Proof. Note that in this case, this implies that X is strictly $\mathcal{O}$-compact over $\operatorname{Spec}(A)$ and thus $\mathcal{O}_X$ is a compact object in $\operatorname{QCoh}(X)$. Consider the following pull-back square:

$$\begin{tikzcd}
X_B \arrow[r, "f'"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec}(B) \arrow[d, "q"] \\
X \arrow[r, "f"'] & \operatorname{Spec}(A)
\end{tikzcd}$$

In this case, since q is affine, p is relatively affine over X, so $X_B \simeq \operatorname{Spec}_X(C)$ for $C = p_*(\mathcal{O}_{X_B}) \in \operatorname{CAlg}(\operatorname{QCoh}(X))$. Moreover, writing $\operatorname{Spec}(B) \simeq \operatorname{Spec}_{\operatorname{Spec}(A)}(B)$ as the relative spectrum of B over A, we also obtain that $X_B \simeq \operatorname{Spec}_X(f^*B)$ by reminder 5.1.4. Therefore, we obtain that $p_*(\mathcal{O}_{X_B}) \simeq f^*B$ as commutative algebras in $\operatorname{QCoh}(X)$, and therefore $\operatorname{QCoh}(X_B) \simeq f^*(B) - \operatorname{Mod}(\operatorname{QCoh}(X))$ by [GR17, Chap. I.3, Prop. 3.3.3]. Under this equivalence, the functor $p^* : \operatorname{QCoh}(X) \to \operatorname{QCoh}(X_B)$ is given by $E \mapsto E \otimes_{\mathcal{O}_X} f^*B$, while $q^* : A - \operatorname{Mod} \to B - \operatorname{Mod}$ is given by $M \mapsto M \otimes_A B$ by definition of q^* in the affine case, and their right adjoints p_* and q_* are the usual forgetful functors.

Let now $E \in \operatorname{QCoh}(X)$. Since $q_* : B - \operatorname{Mod} \to A - \operatorname{Mod}$ is conservative, to check that $q^*f_*E \to f'_*p^*E$ is an equivalence, it suffices to check that $q_*q^*f_*E \to q_*f'_*p^*E$ is an equivalence. The left hand-side is simply $f_*E \otimes_A B$ as an A-module by definition in the affine case. On the right-hand side, since $q \circ f' = f \circ p$, we obtain that $q_*f'_*p^*E \simeq f_*p_*p^*E$, and moreover $p_*p^*E = E \otimes_{\mathcal{O}_X} f^*E$ by the above discussion, so it remains to check that $f_*(E) \otimes_A B \simeq f_*(E \otimes_{\mathcal{O}_X} f^*B)$ as A-modules.

Consider thus the two functors $F, G : A - \operatorname{Mod} \to A - \operatorname{Mod}$ defined by $F(M) = f_*(E) \otimes_A M$ and $G(M) = f_*(E \otimes_{\mathcal{O}_X} f^*M)$. We will show that these two functors coincide, so in particular $F(B) \simeq G(B)$ and thus by definition $f_*(E) \otimes_A B \simeq f_*(E \otimes_{\mathcal{O}_X} f^*B)$ as A-modules, which will conclude the proof. Notice that $F(A) = f_*(E) = G(A)$ by definition. Moreover, since A generates $A - \operatorname{Mod}$ by filtered colimits by [LurieDAG12, Cor. 1.5.12] because $\operatorname{Spec}(A)$ is quasi-compact quasi-separated, it suffices to check that F and G commute with filtered colimits to conclude.

Let us show first that $f_* : \operatorname{QCoh}(X) \to A - \operatorname{Mod}$ commutes with filtered colimits. For this, notice that $f_*(E) \simeq \underline{\operatorname{Hom}}_A(A, f_*E)$ as A-modules. We can view an A as a commutative algebra in spaces, and an A-module is a space equipped with an A-action. In this sense, the underlying space of the above A-module is $\mathbf{Map}_{A-\operatorname{Mod}}(A, f_*E) \simeq \mathbf{Map}_{\operatorname{QCoh}(X)}(\mathcal{O}_X, E)$, and finally since $\mathcal{O}_X$ is compact by assumption in this case, we do get that $f_* : \mathrm{QCoh}(X) \to A - \mathrm{Mod}$ commutes with filtered colimits. Alternatively, we can see that the pull-back functor $f^* : A - \mathrm{Mod} \to \mathrm{QCoh}(X)$ makes $\mathrm{QCoh}(X)$ pseudo-enriched over $A - \mathrm{Mod}$ in the sense of [HA, Def. 4.2.1.25], so for any $E, E' \in \mathrm{QCoh}(X)$, there exists an A-module $\underline{\mathrm{Hom}}_A(E, E')$ as defined in [HA, Def. 4.2.1.28], and by its universal property we can see that $f_*(E) \simeq \underline{\mathrm{Hom}}_A(\mathcal{O}_X, E)$. Once again, since $\mathcal{O}_X$ is compact in $\mathrm{QCoh}(X)$, this implies that f_* commutes with filtered colimits.

Moreover, f^* commutes with all colimits as a left adjoint to f_*, and $\otimes_A$ and $\otimes_{\mathcal{O}_X}$ also commute with all colimits as left adjoints to $\underline{\mathrm{Hom}}_{A-\mathrm{Mod}}$ and $\underline{\mathrm{Hom}}_{\mathrm{QCoh}(X)}$ respectively, so as a composition of these functors both F and G commute with filtered colimits. This concludes the proof. ☐

Proposition 7.3.9. *Let $f : X \to Y$ be an $\mathcal{O}$-compact map of arbitrary derived stacks, and let $f_*^{\mathrm{Mod}} : \mathcal{O}_X - \mathrm{Mod} \to \mathcal{O}_Y - \mathrm{Mod}$ be the pushforward of $\mathcal{O}$-modules as in definition 3.4.6. If $E \in \mathrm{QCoh}(X)$, then $E' := f_*^{\mathrm{Mod}}(E) \in \mathrm{QCoh}(Y)$.*

Proof. In this proof, for any map $g : S \to S'$ of derived stacks, we will distinguish $g_*^{\mathrm{Mod}} : \mathcal{O}_S - \mathrm{Mod} \to \mathcal{O}_{S'} - \mathrm{Mod}$ and $g_*^{\mathrm{QCoh}} : \mathrm{QCoh}(S) \to \mathrm{QCoh}(S')$ the push-forwards of $\mathcal{O}$-modules and quasi-coherent sheaves. Recall from proposition 3.4.10 that both functors formally exist as right adjoints to the pull-back g^*, because it commutes with colimits and the categories are presentable. However, we need not distinguish g^* whether applied to $\mathcal{O}$-modules or quasi-coherent sheaves as they provide the same underlying object. Note that the push-forwards are related by $g_*^{\mathrm{QCoh}} = (-)^{\mathrm{QCoh}} \circ g_*^{\mathrm{Mod}}$, where $(-)^{\mathrm{QCoh}} : \mathcal{O}_{S'} - \mathrm{Mod} \to \mathrm{QCoh}(S)$ is the right adjoint to the inclusion $\mathrm{QCoh}(S') \subset \mathcal{O}_{S'} - \mathrm{Mod}$, which also exists formally because the inclusion commutes with colimits and the categories are presentable.

Starting from $E \in \mathrm{QCoh}(X)$, without any assumptions on f, we will first construct explicitly an $\mathcal{O}_Y$-module E', by a description which only makes sense since $E \in \mathrm{QCoh}(X)$. Note that this E' need not be quasi-coherent in general, even though E was. We will then see that, since f is $\mathcal{O}$-compact, E' will be quasi-coherent. Finally, without the assumption that f is $\mathcal{O}$-compact, we will show that E' satisfies the universal property of $f_*^{\mathrm{Mod}}(E)$. This will conclude the proof.

Construction of E'. As E' is an $\mathcal{O}_Y$-module, recall from definition 3.4.7 that it is the data, for any map $\mathrm{Spec}(A) \to Y$, of an A-module E_u, and for each map $\mathrm{Spec}(B) \xrightarrow{\phi} \mathrm{Spec}(A) \xrightarrow{u} Y$, and letting $v = u \circ \phi : \mathrm{Spec}(B) \to Y$, of a transition map $\phi^* E'_u \to E'_v$ in $B - \mathrm{Mod}$ (there are also higher homotopy data, which we will ignore for this presentation, but one can see that they exist here by noticing that the constructions we make are all canonically ∞-functorial, as they can be presented as compositions of ∞-functors). To say that E' is quasi-coherent is exactly to say that each of these transition maps is an equivalence, which we will only check is true given the assumption that f is $\mathcal{O}$-compact, after giving its general definition with no assumption on f. To make explicit the definitions of E'_u, E'_v, and the map $\phi^* E'_u \to E'v$, we can form the following diagram:

$$\begin{tikzcd}
X_B \arrow[r, "f_B"] \arrow[d, "\phi'"'] \arrow[dd, bend right=60, "v'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Spec}(B) \arrow[d, "\phi"] \arrow[dd, bend left=60, "v"] \\
X_A \arrow[r, "f_A"] \arrow[d, "u'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Spec}(A) \arrow[d, "u"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Then, one can set $E'_u := f^{\mathrm{QCoh}}_{A,*} u'^* E$, which only makes sense because $u'^* E \in \mathrm{QCoh}(X_A)$ since $E \in \mathrm{QCoh}(X)$, and similarly $E'_v := f^{\mathrm{QCoh}}_{B,*} v'^* E$. Notice that $f^{\mathrm{QCoh}}_{B,*} v'^* E = f^{\mathrm{QCoh}}_{B,*} \phi'^* u'^* E$ since $v' = u' \circ \phi'$. The map transition map $\phi^* E'u \to E'_v$ is thus the canonical base-change map $\phi^* f^{\mathrm{QCoh}}_{A,*}(u'^* E) \to f^{\mathrm{QCoh}}_{B,*} \phi'^*(u'^* E)$ at $u'^* E$. Finally, note that in general, this map is not an equivalence.

E' is quasi-coherent when f is $\mathcal{O}$-compact. Assume now and only in this paragraph that f is $\mathcal{O}$-compact. Then, by proposition 7.3.7, f_A is also $\mathcal{O}$-compact, and moreover the top square of the above diagram satisfies exactly the assumptions of lemma 7.3.8, so the base change-formula does hold in this case, and thus the transition map is an equivalence. As a result, in this case, $E' \in \mathrm{QCoh}(Y)$.

E' is equivalent to $f^{\mathrm{Mod}}_*(E)$. To see that E' represents $f^{\mathrm{Mod}}_*(E)$, we can check its universal property. Let $F \in \mathcal{O}_{S'} - \mathrm{Mod}$, we have to show that $\mathbf{Map}_{\mathcal{O}_{S'}-\mathrm{Mod}}(F, E') \simeq \mathbf{Map}_{\mathcal{O}_S-\mathrm{Mod}}(f^* F, E)$. By definition, the data of a map $\phi : F \to E'$ is the data, for any $u : \mathrm{Spec}(A) \to Y$, of a map $\phi_u : F_u \to E'_u = f^{\mathrm{QCoh}}_{A,*} u'^* E$, together with commutativity with their transition maps. Our goal is to see that this data is equivalent to the data of a map $\psi : f_A^* F \to E$ which admits a similar description. Let now $\phi : F \to E'$ be such a map. Notice that $X \simeq \underset{\mathrm{Spec}(A) \to Y}{\mathrm{colim}} X_A$, because pull-backs commute with colimits, and the colimit of $\mathrm{Spec}(A) \to Y$ is $Y = Y$. This means that the data of a map $\mathrm{Spec}(B) \to X$ is the data of a map $\mathrm{Spec}(B) \to \underset{\mathrm{Spec}(A) \to Y}{\mathrm{colim}} X_A$, so by Yoneda and since colimits are computed pointwise in presheaf categories, $X(B) = \underset{\mathrm{Spec}(A) \to Y}{\mathrm{colim}} X_A(B)$. In other words, $\mathbf{Map}_{\mathrm{dSt}}(\mathrm{Spec}(B), X) = \underset{\mathrm{Spec}(A) \to Y}{\mathrm{colim}} \mathbf{Map}_{\mathrm{dSt}}(\mathrm{Spec}(B), X_A)$, so any map $\mathrm{Spec}(B) \to X$ factors through some X_A coming from some $\mathrm{Spec}(A) \to Y$. Let now $v : \mathrm{Spec}(B) \to X$, we want to use ϕ to construct a map $\psi : f_A^* F \to E$, so we want to construct a map $\psi_v : (f_A^* F)_v \to E_v$. Pick any factorization $\mathrm{Spec}(B) \to X_A \to X$ given by some $u : \mathrm{Spec}(A) \to Y$. We can form the following diagram:

$$\begin{tikzcd}
\mathrm{Spec}(B) \arrow[d, "v'"'] \arrow[dd, bend right=60, "v"'] & \\
X_A \arrow[r, "f_A"] \arrow[d, "u'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spec}(A) \arrow[d, "u"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

In the data of ϕ, we have a map $\phi_u : F_u \to f_{A,*}^{\mathrm{QCoh}}(u'^*E)$. By adjunction, this yields a map $\phi_u^\flat : f_A^* F_u \to u'^*E \in \mathcal{O}_{X_A} - \mathrm{Mod}$. Pulling-back along v'^*, we thus obtain a map $v'^* f_A^* F_u \to v'^* u'^* E \in \mathcal{O}_{\mathrm{Spec}(B)} - \mathrm{Mod}$. Notice that by commutativity of the diagram, this is equivalent to a map $(f \circ v)^* F_u \to v^* E \in \mathcal{O}_{\mathrm{Spec}(B)} - \mathrm{Mod}$. Finally, evaluating at the identity $\mathrm{Spec}(B) = \mathrm{Spec}(B)$, we obtain a map of B-modules $\psi_v : F_{f \circ v} \to E_v$. By definition 3.4.6, after unfolding, we can see that $F_{f \circ v} = (f_A^* F)_v$. We omit the verification that the collection of maps ψ_v that we have just provided is indeed a map of $\mathcal{O}_X$-modules, but it is straightforward. A similar construction can be done in the other direction to produce a map ϕ from the data of a map ψ, and finally one can check that these associations are inverse to each other up to homotopy. We thus obtain that the $\mathcal{O}_Y$-module E' as we have defined above does indeed satisfy the universal property of $f_*^{\mathrm{Mod}}(E)$, so they are canonically equivalent. This concludes the proof. ☐

Proposition 7.3.10 (Arbitrary base-change)**.** *Let $f : X \to Y$ be an $\mathcal{O}$-compact map of derived stacks, and $q : V \to Y$ be an arbitrary map of derived stacks. Let $f' : q^*V \to V$ be the pull-back of f, and $p : q^*V \to X$ be the pull-back of q. Then, for any $E \in \mathrm{QCoh}(X)$, the canonical base-change map $q^* f_* E \to f'_* p^* E \in \mathrm{QCoh}(V)$ is an equivalence, so the base-change formula holds.*

Proof. Note that by proposition 7.3.7, f' is also $\mathcal{O}$-compact. As such, by proposition 7.3.9, both f_* and f'_* are simply given as a restriction of the push-forward of $\mathcal{O}$-modules on quasi-coherent sheaves, which remain quasi-coherent in this case. Moreover, as pull-back functors, both q^* and p^* also are the restrictions of the pull-back of $\mathcal{O}$-modules to quasi-coherent sheaves. Therefore, we are reduced to checking the base-change formula for $\mathcal{O}$-modules on the big site, which always holds simply by unfolding the definitions. ☐

Remark 7.3.11. An immediate consequence of base-change is that, for $f : X \to Y$ an $\mathcal{O}$-compact map, and for $E \in \mathrm{Perf}(X)$, we can see that $f_*(E)$ is again perfect. This is because this object is perfect if and only if, for any $u : \mathrm{Spec}(A) \to Y$, $u^* f_*(E)$ is perfect, but by the base-change in proposition 7.3.10, this is equivalent to $f_{A,*}(u'^*E)$, where $X_A = X \times_Y \mathrm{Spec}(A)$ is the pull-back, and $f_A : X_A \to \mathrm{Spec}(A)$ and $u' : X_A \to X$ are the canonical maps. Finally, this latter object is indeed perfect because u'^* preserves perfect as it is a pull-back, and $f_{A,*}$ preserves perfects by the assumption that f is $\mathcal{O}$-compact. This is the fundamental reason to add the assumption of preservation of perfects in the definition of $\mathcal{O}$-compactness.

Proposition 7.3.12. *Let $f : X \to Y$ be a proper-schematic and local complete intersection map of derived stacks. Then f is $\mathcal{O}$-compact.*

Proof. Let $\mathrm{Spec}(A) \to Y$ and consider $f_A : X_A := X \times_Y \mathrm{Spec}(A) \to \mathrm{Spec}(A)$. By assumption, X_A is a proper derived scheme over $\mathrm{Spec}(A)$, and moreover f_A is local complete intersection as a pull-back of f. Thus, by [Toë12, Thm. 2.1], f_A preserves perfect complexes, and it remains to check that $\mathcal{O}_{X_A}$ is compact in $\mathrm{QCoh}(X_A)$. More generally, by [LurieDAG12, Cor. 1.5.12], quasi-compact quasi-separated derived stacks S are perfect in the sense that $\mathrm{QCoh}(S)$ is compactly generated and its objects are compact if and only if they are perfect (dualizable). In particular, X_A is quasi-compact quasi-separated since it is proper, and $\mathcal{O}_{X_A}$ is perfect as it is the unit of the monoidal structure, thus it is compact. ☐

Remark 7.3.13. Let $f : X \to Y$ be an $\mathcal{O}$-compact map of derived stacks. Suppose we are given a map $[X] : f_*\mathcal{O}_X \to \mathcal{O}_Y[-d] \in \mathrm{QCoh}(Y)$, and a point $u : \mathrm{Spec}(A) \to Y$. Then we can consider the following pull-back:

$$\begin{tikzcd}
X_A \arrow[r, "f_A"] \arrow[d, "u'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spec}(A) \arrow[d, "u"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

By proposition 7.3.10, in particular we have that $u^*f_*\mathcal{O}_X \simeq f_{A,*}u'^*\mathcal{O}_X \simeq f_{A,*}\mathcal{O}_{X_A}$. As such, if we set $[X]_A := u^*[X] : u^*f_*\mathcal{O}_X \to u^*\mathcal{O}_Y[-d]$, by the above equivalence and since $u^*\mathcal{O}_Y[-d] \simeq A[-d]$, we can identify it as a map $[X]_A : f_{A,*}\mathcal{O}_{X_A} \to A[-d]$.

Definition 7.3.14. *Let $f : X \to Y$ be a an $\mathcal{O}$-compact map of derived stacks. An* $\mathcal{O}$-orientation of degree d on f *is the data of a map:*

$$[X] : f_*\mathcal{O}_X \to \mathcal{O}_Y[-d] \in \mathrm{QCoh}(Y)$$

such that, for any map $u : \mathrm{Spec}(A) \to Y$, with pull-back $f_A : X_A = X \times_Y \mathrm{Spec}(A) \to \mathrm{Spec}(A)$, and any $E \in \mathrm{Perf}(X_A)$, the following induced map is a perfect pairing in $A\!-\!Mod$:

$$f_{A,*}(E) \otimes f_{A,*}(E^\vee) \xrightarrow{f_{A,*} \text{ lax mon.}} f_{A,*}(E \otimes E^\vee) \xrightarrow{f_{A,*}(v_E)} f_{A,*}(\mathcal{O}_{X_A}) \xrightarrow{[X]_A} A[-d]$$

where $[X]_A = u^*[X]$ as above, and $v_E : E \otimes E^\vee \to \mathcal{O}_X$ is the canonical perfect pairing of E and its dual. To say that the above pairing is perfect means that it induces an equivalence:

$$f_{A,*}(E) \simeq f_{A,+}(E)[-d]$$

where, as in definition 6.1.6, $f_{A,+}(E) := f_*(E^\vee)^\vee$.

Remark 7.3.15. In the above situation, for $E \in \mathrm{Perf}(X)$, we can also form a canonical pairing:

$$f_*(E) \otimes f_*(E^\vee) \xrightarrow{f_* \text{ lax mon.}} f_*(E \otimes E^\vee) \xrightarrow{f_*(v_E)} f_*(\mathcal{O}_X) \xrightarrow{[X]} \mathcal{O}_Y[-d]$$

To check that it is perfect, which makes sense since f_* preserves perfects by remark 7.3.11, it suffices to check that the pull-back of this pairing along any $u : \mathrm{Spec}(A) \to Y$ is perfect, which is true exactly by the assumption that $[X]$ is an $\mathcal{O}$-orientation of degree d and base-change. Therefore, we also have that $f_*(E) \simeq f_+(E)[-d]$ for $E \in \mathrm{QCoh}(X)$.

Proposition 7.3.16. *Let $f : X \to Y$ be an $\mathcal{O}$-compact map equipped with $[X]$ an $\mathcal{O}$-orientation of degree d. Let $q : V \to Y$ be any map of derived stacks, so the pull-back $f' : q^*X \to V$ is $\mathcal{O}$-compact by proposition 7.3.7. In this case, $q^*[X]$ is an $\mathcal{O}$-orientation of degree d on f'.*

Proof. As in remark 7.3.13, we can identify the source and target of $q^*[X]$ to view it as a map $q^*[X] : f'_*\mathcal{O}_{q^*X} \to \mathcal{O}_V$. We will now denote it as $[q^*X]$, and we will show that it is an $\mathcal{O}$-orientation of degree d on f'. Let thus $u : \mathrm{Spec}(A) \to V$ be a point, we can form the following diagram:

$$\begin{tikzcd}
q^*X_A \arrow[r, "f'_A"] \arrow[d, "u'"'] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Spec}(A) \arrow[d, "u"] \\
q^*X \arrow[r, "f'"] \arrow[d, "q'"'] \arrow[dr, phantom, "\lrcorner"] & V \arrow[d, "q"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

By pasting, the map f'_A coincides with the map f_A of definition 7.3.14, and moreover we have that $[q^*X]_A = u^*[q^*X] = u^*q^*[X] = (q \circ u)^*[X]$. Therefore, for any $E \in \mathrm{Perf}(q^*X_A)$, the canonical pairing associated to $[q^*X]$ coincides by definition with the canonical pairing associated to $[X]$ at the point $q \circ u : \mathrm{Spec}(A) \to Y$, and therefore it is a perfect pairing by assumption on $[X]$. ☐

Remark 7.3.17. In this chapter, so far we have considered maps $Z \to X$ which are locally of finite presentation and admit a relative cotangent complex $\mathbb{L}_{Z/X}$, without the need to restrict ourselves further. In the following theorem, we will need the push-forward of derived foliations to be defined, so we will further restrict ourselves to the case of a relative derived Deligne-Mumford stack locally of finite presentation, so that theorem 6.2.2 applies and a pushforward of derived foliations exist.

Theorem 7.3.18 (Theorem 2.3.2). *Let $Z \to X$ be a relative derived Deligne-Mumford stack locally of finite presentation, and $f : X \to Y$ a proper-schematic, local complete intersection, and flat morphism of derived stacks. Assume that $f_*Z \to Y$ is a relative derived Artin stack locally of finite presentation. Let f be equipped with $[X]$ an $\mathcal{O}$-orientation* *of degree d, and let $\mathcal{F} \in \mathcal{F}ol(Z/X)$ be a derived foliation on Z relative to X equipped with an n-shifted symplectic form $\omega \in \mathbb{S}ymp(\mathcal{F}, n)$. Then the push-forward derived foliation $f_*\mathcal{F} \in \mathcal{F}ol(f_*Z/Y)$ inherits an $(n-d)$-shifted symplectic form $f_*\omega \in \mathbb{S}ymp(f_*\mathcal{F}, n-d)$.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
Z \arrow[dr] & f^*f_*Z \arrow[l, "ev"'] \arrow[r, "\pi"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & f_*Z \arrow[d, "q"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

We will start by constructing canonical $\mathcal{H}$-equivariant maps $f_*\mathbf{A}^p_{Z/X}[n] \to \mathbf{A}^p_{f_*Z/Y}[n-d]$, where note that $f_*\mathbf{A}^p_{Z/X}[n]$ inherits a canonical $\mathcal{H}$-action by monoidality of f_* since it commutes with limits as a right adjoint. By definition of $\mathbf{A}^p_{-/-}[n]$ and more specifically by remark 7.2.14, and by the formula provided by corollary 6.1.11 (without preserving the $\mathbb{G}_m$-actions in this case, so in fact we are applying directly lemma 6.1.8 and reminder 5.1.4), we obtain that, as derived stacks over f_*Z we have canonical identifications:

$$\mathbf{A}^p_{f_*Z/Y}[n-d] \simeq \mathbb{V}_{f_*Z}(\mathcal{O}_{f_*Z}[d+p-n])$$
$$f_*\mathbf{A}^p_{Z/X}[n] \simeq \mathbb{V}_{f_*Z}(\pi_+(ev^*\mathcal{O}_Z[p-n]))$$

Therefore, it suffices to provide canonical maps of graded mixed modules $\mathcal{O}_{f_*Z}(p)[d+p-n] \to \pi_+(ev^*\mathcal{O}_Z(p)[p-n])$, as its $\mathbb{V}_{f_*Z}$ will thus be $\mathcal{H}$-equivariant because in this case $\mathcal{H}$-equivariance simply means that the map respects the graded mixed structures of their defining algebras on f_*Z. Since such maps are concentrated in weight p, and the mixed differentials are 0 on both sides, it is equivalent to provide maps of the following form in QCoh(f_*Z):

$$\mathcal{O}_{f_*Z}[d+p-n] \to \pi_+(ev^*\mathcal{O}_Z[p-n])$$

or, after shifting by $[-(p-n)]$ and dualization, it suffices to provide a single map of the following form in QCoh(f_*Z):

$$\pi_*(ev^*\mathcal{O}_Z) \to \mathcal{O}_{f_*Z}[-d]$$

Notice that $ev^*\mathcal{O}_Z \simeq \mathcal{O}_{f^*f_*Z}$, so finally it suffices to provide a canonical map:

$$\pi_*(\mathcal{O}_{f^*f_*Z}) \to \mathcal{O}_{f_*Z}[-d]$$

We will set this map to be $q^*[X]$ the pull-back of $[X]$ the $\mathcal{O}$-orientation of degree d on f, which is thus an $\mathcal{O}$-orientation of degree d on π by props. 7.3.12, 7.3.7 and 7.3.16.

Consider now the $\mathcal{H}$-equivariant map $\mathbb{V}(\mathcal{F}) \to \mathbf{A}^2_{Z/X}[n]$ encoding the data of $\omega \in \mathbb{S}ymp(\mathcal{F}, n)$. Applying the Weil restriction $f_* : \mathrm{dSt}/X \to \mathrm{dSt}/Y$ to this map and composing with the above map, we obtain an $\mathcal{H}$-equivariant map:

$$\mathbb{V}(f_{*,\mathcal{F}ol}\mathcal{F}) = f_*\mathbb{V}(\mathcal{F}) \to f_*\mathbf{A}^2_{Z/X}[n] \to \mathbf{A}^p_{f_*Z/Y}[n-d]$$

and thus we obtain an $(n-d)$-shifted closed 2-form $f_*\omega \in \mathcal{A}^{2,cl}(f_*\mathcal{F}, n-d)$.

To see that it is non-degenerate, let us unfold its definition. Starting from the underlying non-degenerate n-shifted 2-form of ω, expressed as a map $\omega : \mathcal{O}_Z[-n] \to \wedge^2\mathbb{L}_{\mathcal{F}}$, firstly compute $\omega' := \pi_+ ev^*\omega : \pi_+\mathcal{O}_{f^*f_*Z}[-n] \simeq \pi_+ ev^*\mathcal{O}_Z[-n] \to \wedge^2\pi_+ ev^*\mathbb{L}_{\mathcal{F}}$, and then compose it with the map $\mathcal{O}_{f_*Z}[d-n] \to \pi_+\mathcal{O}_{f^*f_*Z}[n]$ as constructed above. This yields the $(n-d)$-shifted 2-form $f_*\omega : \mathcal{O}_{f_*Z}[d-n] \to \wedge^2\pi_+ ev^*\mathbb{L}_{\mathcal{F}}$. By duality, this yields a pairing: $\pi_* ev^*\mathbb{T}_{\mathcal{F}} \otimes \pi_* ev^*\mathbb{T}_{\mathcal{F}} \to \mathcal{O}_{f_*Z}[n-d]$. Equivalently, this pairing can be obtained in the following way. Consider the canonical pairing:

$$\begin{aligned}
&\pi_* ev^*\mathbb{T}_{\mathcal{F}} \otimes \pi_* ev^*\mathbb{L}_{\mathcal{F}} \\
\xrightarrow{\pi_* ev^* \text{ lax mon.}}\ &\pi_* ev^*(\mathbb{T}_{\mathcal{F}} \otimes \mathbb{L}_{\mathcal{F}}) \\
\xrightarrow{\pi_* ev^*(v_{ev^*\mathbb{T}_{\mathcal{F}}})}\ &\pi_* ev^*\mathcal{O}_Z \\
\simeq\ &\pi_*(\mathcal{O}_{f^*f_*Z}) \\
\xrightarrow{q^*[X]}\ &\mathcal{O}_{f_*Z}[-d]
\end{aligned}$$

By assumption, $[X]$ is an $\mathcal{O}$-orientation, so $q^*[X]$ is an $\mathcal{O}$-orientation too by proposition 7.3.16, and thus this pairing is perfect. Moreover, using $\omega : \mathbb{T}_{\mathcal{F}} \xrightarrow{\sim} \mathbb{L}_{\mathcal{F}}[n]$, or equivalently $\mathbb{L}_{\mathcal{F}} \simeq \mathbb{T}_{\mathcal{F}}[-n]$, the above pairing induces the following pairing, which exactly corresponds to the one coming from the underlying 2-form of $f_*\omega$:

$$\pi_* ev^*\mathbb{T}_{\mathcal{F}} \otimes \pi_* ev^*\mathbb{T}_{\mathcal{F}}[-n] \to \mathcal{O}_{f_*Z}[-d]$$

or equivalently this is a pairing of the form:

$$\pi_* ev^*\mathbb{T}_{\mathcal{F}} \otimes \pi_* ev^*\mathbb{T}_{\mathcal{F}} \to \mathcal{O}_{f_*Z}[n-d]$$

Since the previous pairing is perfect, and the map $\omega : \mathbb{T}_{\mathcal{F}} \xrightarrow{\sim} \mathbb{L}_{\mathcal{F}}[n]$ is an equivalence, this induced pairing is perfect too, which implies that the form $f_*\omega$ induces an equivalence $\pi_*(ev^*\mathbb{T}_{\mathcal{F}}) \simeq \pi_+(ev^*\mathbb{L}_{\mathcal{F}})[n-d]$. By remark 6.2.6, we can identify $\mathbb{T}_{f_{*,\mathcal{F}ol}\mathcal{F}} \simeq \pi_*(ev^*\mathbb{T}_{\mathcal{F}})$, and $\mathbb{L}_{f_{*,\mathcal{F}ol}\mathcal{F}} \simeq \pi_+ ev^*\mathbb{L}_{\mathcal{F}}$, so the previous equivalence is in fact exactly the data of an equivalence

$$\mathbb{T}_{f_{*,\mathcal{F}ol}\mathcal{F}} \simeq \mathbb{L}_{f_{*,\mathcal{F}ol}\mathcal{F}}[n-d]$$

i.e. the $(n-d)$-shifted 2-form underlying $f_*\omega$ is non-degenerate, so we have constructed a canonical element

$$f_*\omega \in \mathbb{S}ymp(f_{*,\mathcal{F}ol}\mathcal{F}, n-d)$$

□

CHAPTER 8

APPLICATIONS

Let us now prove and discuss some applications of this thesis.

8.1 FOLIATIONS FOR GROMOV-WITTEN THEORY

Let us start by providing more details to example 2.2.4.

Reminder 8.1.1. As a first application, let us consider the moduli stacks of pre-stable and stable curves of genus g with n marked points. We take the following constructions and properties from [STV15, Section 2.2].

- There is a derived stack $\mathbb{R}\mathbf{M}^{pre}_{g,n}$, called the derived moduli stack of pre-stable curves of genus g with n marked points, that comes equipped with a universal family $\mathbb{R}\mathcal{C}^{pre}_{g,n}$ defined over $\mathbb{R}\mathbf{M}^{pre}_{g,n}$, whose classical truncations coincide with the usual $\mathbf{M}_{g,n}$ and $\mathcal{C}_{g,n}$ respectively.
- As such, for X a smooth (possibly derived) projective scheme, we can define the derived stack $\mathbb{R}\mathbf{M}^{pre}_{g,n}(X)$ as the following derived mapping stack, canonically defined over $\mathbb{R}\mathbf{M}^{pre}_{g,n}$:
$$\mathbb{R}\mathbf{M}^{pre}_{g,n}(X) := \underline{\mathbf{Map}}_{\mathbb{R}\mathbf{M}^{pre}_{g,n}}(\mathbb{R}\mathcal{C}^{pre}_{g,n}, X \times \mathbb{R}\mathbf{M}^{pre}_{g,n})$$
Note that points of this stack morally correspond to families of pre-stable curves of genus g with n marked points in X. Its truncation recovers the usual non-derived moduli stack $\mathbf{M}^{pre}_{g,n}(X)$ of pre-stable curves of genus g with n marked points. As such, it is Artin locally of finite presentation, and thus $\mathbb{R}\mathbf{M}^{pre}_{g,n}(X)$ is derived Artin locally of finite presentation too. See [HR15, Cor. 2.4(ii), Rmk. 2.5] and [Beh97, Prop. 2] to obtain this last fact on the classical stack.
- Moreover, $\mathbb{R}\mathbf{M}^{pre}_{g,n}(X)$ contains an open sub-stack $\mathbb{R}\overline{\mathbf{M}}_{g,n}(X;\beta)$, which constitutes similarly the derived moduli stack of families of stable curves of genus g with n marked points and of class β on X, and whose classical truncation also coincides with the usual $\overline{\mathbf{M}}_{g,n}(X;\beta)$ non-derived moduli stack of such objects. As it is an open substack of the previous one, it is also Artin locally of finite presentation, and thus $\mathbb{R}\overline{\mathbf{M}}_{g,n}(X;\beta)$ is too.
- Finally, note that the map $\mathbb{R}\mathcal{C}^{pre}_{g,n} \to \mathbb{R}\mathbf{M}^{pre}_{g,n}$ is proper-schematic (i.e. representable by proper derived schemes), flat, and local complete intersection.

Corollary 8.1.2. *Let X be a smooth (possibly derived) projective scheme equipped with a derived foliation. Then $\mathbb{R}\mathbf{M}^{pre}_{g,n}(X)$ and $\mathbb{R}\overline{\mathbf{M}}_{g,n}(X;\beta)$ inherit derived foliations.*

Proof. The derived foliation on X induces one on $X \times \mathbb{R}\mathbf{M}^{pre}_{g,n}$ by pull-back along the projection to X. The derived foliation on $\mathbb{R}\mathbf{M}^{pre}_{g,n}(X)$ is then obtained as a direct application of theorem 2.2.1 to this derived foliation, as all the conditions of the theorem are satisfied in this case by the above reminder. It also induces one on $\mathbb{R}\overline{\mathbf{M}}_{g,n}(X;\beta)$ by restriction (i.e. pull-back) along the inclusion. ☐

Remark 8.1.3. In the study of Gromov-Witten invariants, where the new data of a derived foliation could lead to the definition of new invariants relative to it, the space $\mathbb{R}\overline{\mathbf{M}}_{g,n}(X;\beta)$ is typically considered when X is a smooth projective scheme, which is why this result was stated in this case. While our construction would also apply when X is a derived Deligne-Mumford stack locally of finite presentation, Gromov-Witten invariants on such an X can involve different definitions and constructions, which we will not discuss in this thesis.

8.2 FOLIATIONS ON HILBERT SCHEMES

We will now provide more details to example 2.2.5.

Reminder 8.2.1. As a second application, let us now rephrase the construction of $\mathbb{R}\mathbf{Hilb}^{lci}(Y)$ from [CK02] in terms of mapping stacks. See also the last example of [Toë14a, §3.2] which provides exactly the same construction in a slightly different presentation.

Let $Sch_{prop,flat,lci} : \mathrm{dAff}^{\mathrm{op}} \to \infty - Grpd$ be the functor defined by:

$$Sch_{prop,flat,lci}(S) := \{\text{derived schemes } Z \to S \text{ that are proper, flat and lci over } S\}$$

This functor is a derived stack, and moreover it is equipped with a universal object $\mathcal{S} \to Sch_{prop,flat,lci}$ which is a relative derived scheme, proper-schematic, flat, and local complete intersection. Let Y be a derived Deligne-Mumford stack locally of finite presentation, and consider the derived stack:

$$\underline{\mathbf{Map}}_{Sch_{prop,flat,lci}}(\mathcal{S}, Y \times Sch_{prop,flat,lci})$$

As mentioned in the last example of [Toë14a, §3.2], this is a derived Artin stack locally of finite presentation over $Sch_{prop,flat,lci}$. Note that if Y is equipped with a derived foliation, following a proof analogous to corollary 8.1.2, since the conditions of theorem 2.2.1 also apply in this case, this derived stack thus inherits a derived foliation. Moreover, for a derived affine S, the S-points of this derived stack are equivalent to the data of a derived S-scheme Z which is proper, flat, and local complete intersection over S, and moreover equipped with a map $Z \to Y$. As such, it contains $\mathbb{R}\mathbf{Hilb}^{lci}(Y)$, the derived Hilbert stack, whose S-points consist in those $Z \to S$ such that the equipped map $Z \to Y$ is moreover a closed immersion. This substack also inherits a derived foliation by restriction.

Corollary 8.2.2. *Let Y be a derived Deligne-Mumford stack locally of finite presentation, equipped with a derived foliation. Then the derived moduli stack of closed derived subschemes of Y that are flat, proper, and local complete intersection over their base, denoted by $\mathbb{R}\mathbf{Hilb}^{lci}(Y)$, inherits a derived foliation.*

As mentioned in the introduction, we expect integral points of this inherited derived foliation in a sense analogous to [CC09, Def. 1.9.2(1)], i.e. the locus where its cotangent vanishes, to correspond to closed derived subschemes of Y contained in the algebraic leaves of its derived foliation. A rigorous statement and proof of this result is outside the scope of this thesis, but will be studied instead in a future work, as mentioned in the next chapter.

More generally, these two examples of applications can be thought of as illustrations of the transfer of derived foliations to a wide class of derived moduli stacks.

CHAPTER 9

ONGOING AND FUTURE WORK

Let us now talk about ongoing and future work following this thesis. The first section of this chapter concerns the ongoing work, while the next sections concern future work.

9.1 EXTENSION BEYOND THE DELIGNE-MUMFORD CASE

Firstly, in an ongoing work, we expect the results of this thesis to extend to the Artin case beyond the Deligne-Mumford case. In fact, in an ongoing work, we expect the right assumption to be that $Z \to X$ is any map of derived stacks which admits a relative cotangent $\mathbb{L}_{Z/X} \in \mathrm{QCoh}(Z)$ which is moreover perfect, and we will say that the map $Z \to X$ admits a perfect relative cotangent when it is the case. Note that this is the case for example when $Z \to X$ is a relative derived Artin stack locally of finite presentation, but we expect our results to hold even without the derived Artin assumption. We can also hope to extend the results when the map $f : X \to Y$ along which we push forward is not representable, and in particular it seems that the condition of $\mathcal{O}$-compactness of definition 7.3.5 is sufficient. Note similarly that this condition is implied by the current proper-schematic and lci assumption by proposition 7.3.12. As such, we expect to prove the following theorems.

Expected result 9.1.1. *Let $Z \to X$ be a map of derived stacks which admits a perfect relative cotangent, and $f : X \to Y$ an $\mathcal{O}$-compact map of derived stacks. Then there is a push-forward functor:*

$$f_{*,\mathcal{F}ol} : \mathcal{F}ol(Z/X) \to \mathcal{F}ol(f_*Z/Y)$$

Expected result 9.1.2. *Let $Z \to X$ be a map of derived stacks which admits a perfect relative cotangent, and $f : X \to Y$ an $\mathcal{O}$-compact map of derived stacks equipped with an $\mathcal{O}$-orientation of degree d. If $\mathcal{F}$ is an n-shifted symplectic derived foliation on Z relative to X, then the derived foliation on f_*Z relative to Y provided by $f_{*,\mathcal{F}ol}$ is equipped with an $(n-d)$-shifted symplectic structure.*

Expected result 9.1.3. *In particular, if X is an $\mathcal{O}$-compact, d-oriented derived stack over a base S, and Y is a derived stack over S which admits a perfect relative cotangent over S, and is equipped with an n-shifted symplectic derived foliation, then the induced derived foliation on the derived mapping stack $\underline{\mathbf{Map}}_S(X, Y)$ has an $(n-d)$-shifted symplectic structure.*

Remark 9.1.4. In the above expected results, not only were the results extended from the Deligne-Mumford to this more general case, and the representability of f was lifted as well, but it is important to note that in this situation the flatness of f should be able to be removed. This is because in this thesis the flatness of f is only introduced to allow the Weil-restriction along f to preserve some Deligne-Mumford stacks, but it can be removed if we can extend the results beyond the Deligne-Mumford case. However it means that in this situation the derived stack $f_*Z \to Y$ might lose some geometric representability even if $Z \to X$ did have some, e.g. if it was relative derived Artin, and thus $f_*Z \to Y$ will be less well-behaved.

We will now discuss future work.

9.2 Comparison with Poisson Structures

Using the results of [Tom25], we expect to link shifted Poisson structures, as defined in [CPTVV], to shifted symplectic derived foliations. Therefore, we also expect the mapping theorem for shifted symplectic derived foliations to be related to the mapping theorem for shifted Poisson structures proved in [Tom26, Thm. A]. However, it is not clear if, as in differential geometry, Poisson structures are really equivalent to shifted symplectic derived foliations, which we would like to investigate in a future joint work with Nikola Tomić.

9.3 Geometric Quantization

Together, the above results are expected to have applications, among others, in higher geometric quantization. For example, see [Saf23], which constructs a geometric quantization when given the input data of a Lagrangian fibration. This corresponds in our setting to a Lagrangian derived foliation which is integrable, while our results already extend to the non-integrable case. We would like to investigate if the geometric quantization constructed in this paper also extends to non-integrable Lagrangian derived foliations.

9.4 Exterior Differential Systems

In a different direction, we plan to study exterior differential systems. In differential geometry, exterior differential systems can be seen as generalizations of foliations. Foliations correspond to partial differential equation systems that only involve 1-forms, while exterior differential systems can involve p-forms for multiple $p \geq 1$, with finitely many generators and relations. In this context, obstructions to the integrability of an exterior differential system are measured by Spencer cohomology, see [BCGGG91]. In a work in preparation, Eyssidieux and Kwon study exterior differential systems in classical algebraic geometry, and plan to express the moduli space of their solutions as a subspace of the Hilbert scheme of closed subschemes.

In a future work, we will study the definition of a derived enhancement of exterior differential systems. Essentially, for a map of derived stacks $Z \to X$, an exterior differential system on Z relative to X is a graded mixed algebra $F \in \epsilon - cdga^{gr}(Z/X)$ which is a compact object in this category, a condition which expresses the fact that F has a finite presentation in some sense. We plan to prove a mapping theorem for such structures by extending the above push-forward of derived foliations to such algebras, and investigate the connection between the derived enhancement of the moduli space of their solutions and Spencer cohomology. In particular, a mapping theorem for exterior differential systems should allow us to induce an exterior differential system on the derived Hilbert scheme constructed above in reminder 8.2.1, and we should be able to relate integral points of this system to the moduli space of solutions of the initial system. Moreover, we can study the formal integration of such systems in derived geometry, possibly using the formal integration results from [BMN25b].

Chapter A

Beck-Chevalley Conditions

In section 5.2, we use the formalism of Beck-Chevalley squares to obtain the equivalence between derived foliations and perfect linear stacks in the affine case. This formalism originates from base-change formulas in algebraic geometry, which appeared first in [EGA III.2, (6.1.1)], which are particular instances of Beck-Chevalley conditions that also reappear here in section A.2. The term itself was only coined later by Bénabou and Roubaud in 1970, which discussed the property in [BR70], referring to Beck and Chevalley who also worked on the notion around the same time without publishing about it. In the modern context of ∞-category theory, the notion was defined by Lurie in [HA, Def. 4.7.4.13], which he also refers to as adjointable squares. In this appendix, we will recall these definitions and basic properties about them in section A.1, and their appearance in our work in derived algebraic geometry as well as sufficient conditions for them to hold in section A.2.

A.1 Generalities on Beck-Chevalley Conditions

In this section, we will focus on recalling the definition of the Beck-Chevalley condition in ∞-category theory and results following [HA, Def. 4.7.4.13].

Reminder A.1.1. Consider a diagram of ∞-categories of the following form (which is not assumed to be commutative in any way):

$$\begin{tikzcd}
\mathcal{C}' \arrow[d, "U"'] & \mathcal{D}' \arrow[l, shift right, "L'"'] \arrow[l, phantom, "\perp"] \arrow[d, "V"] \\
\mathcal{C} \arrow[r, shift right, "R"'] & \mathcal{D} \arrow[l, shift right, "L"'] \arrow[l, phantom, "\perp"]
\end{tikzcd}$$

with $\mathcal{C}' \xrightarrow{R'} \mathcal{D}'$,

where $L \dashv R$ and $L' \dashv R'$ are adjunctions. We will moreover denote by η (resp. η') the unit $Id_{\mathcal{D}} \implies L \circ R$ of the adjunction $L \dashv R$ (resp. $L' \dashv R'$), and ϵ (resp. ϵ') its co-unit.

- Natural transformations of the form $V \circ R' \implies R \circ U$ (resp. $L \circ V \implies U \circ L'$) will be called right transformations (resp. left transformations).

- Given a right transformation $\phi : V \circ R' \implies R \circ U$, we can construct the following left transformation: $\phi^L := L \circ V \implies L \circ V \circ R' \circ L' \implies L \circ R \circ U \circ L' \implies U \circ L'$, where the first map is induced by the unit of $L' \dashv R'$, the second is induced by ϕ, and the last is induced by the co-unit of $L \dashv R$. More precisely, $\phi^L := (\epsilon \star (U \circ L')) \circ (L \star \phi \star L') \circ ((L \circ V) \star \eta')$. We will call ϕ^L the left adjoint of ϕ.
- Conversely, given a left transformation $\psi : L \circ V \implies U \circ L'$, we can construct a right transformation $\psi^R : V \circ R' \implies R \circ U$ using this time the unit of $L \dashv R$, ψ, and the co-unit of $L' \dashv R'$. More precisely, $\psi^R := ((R \circ U) \star \epsilon') \circ (R \star \psi \star R') \circ (\eta \star (V \circ R'))$. We will call ψ^R the right adjoint of ψ.
- The operations $-^L$ and $-^R$ are inverse to each other (by the triangle identities of η, ϵ and η', ϵ') and promote to an equivalence between the spaces of right and left transformations: $-^L : Nat(V \circ R', R \circ U) \simeq Nat(L \circ V, U \circ L') : -^R$.
- Starting from the square formed by U, V and R, R' only (resp. L, L' only) equipped with a given right transformation (resp. left transformation), the data of the square formed by U, V and L, L' (resp. R, R') equipped with the left adjoint (resp. right adjoint) of the transformation is called the transposed diagram.

Note that neither $-^L$ nor $-^R$ necessarily preserve natural equivalences in general, so the transpose of a transformation which was an equivalence might not still be an equivalence. This motivates the definition of the Beck-Chevalley condition (as in [HA, Rmk. 4.7.4.15]).

Reminder A.1.2. In the context of reminder A.1.1, given a left or a right transformation, we say that it satisfies the Beck-Chevalley condition if it is an equivalence and its adjoint also is an equivalence. The data of a square equipped with such a transformation will be called a Beck-Chevalley square.

In this section we will use in particular the following properties of Beck-Chevalley squares also present in [HA] (resp. Def. 4.7.4.16 and Cor. 4.7.4.18).

Proposition A.1.3. *There is a canonical ∞-category $\mathcal{BC}$ such that:*

- *Its objects are quadruples $(\mathcal{C}, \mathcal{D}, R, L)$ where $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, and $R : \mathcal{C} \to \mathcal{D}$ and $L : \mathcal{D} \to \mathcal{C}$ form an adjunction $L \dashv R$.*
- *Given two objects $(\mathcal{C}', \mathcal{D}', R', L')$ and $(\mathcal{C}, \mathcal{D}, R, L)$, a 1-morphism from the first to the second is the data of Beck-Chevalley square:*

$$\begin{tikzcd}
\mathcal{C}' \arrow[d, "U"'] \arrow[r, shift right, "R'"'] \arrow[r, phantom, "\perp"] & \mathcal{D}' \arrow[l, shift right, "L'"'] \arrow[d, "V"] \\
\mathcal{C} \arrow[r, shift right, "R"'] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift right, "L"']
\end{tikzcd}$$

In particular, note that the fact it is an ∞-category implies that such squares can be composed. More precisely, consider two consecutive Beck-Chevalley squares:

$$\begin{tikzcd}
\mathcal{C}'' \arrow[r, shift right, "R''"'] \arrow[r, phantom, "\perp"] \arrow[d, "U'"'] & \mathcal{D}'' \arrow[l, shift right, "L''"'] \arrow[d, "V'"] \\
\mathcal{C}' \arrow[r, shift right, "R'"'] \arrow[r, phantom, "\perp"] \arrow[d, "U"'] & \mathcal{D}' \arrow[l, shift right, "L'"'] \arrow[d, "V"] \\
\mathcal{C} \arrow[r, shift right, "R"'] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift right, "L"']
\end{tikzcd}$$

where we call ϕ the Beck-Chevalley right transformation for the bottom square, and ψ its adjoint (resp. ϕ' and ψ' for the top square). Then the exterior pasting square also is Beck-Chevalley when equipped with the right transformation $\Phi := (\phi \star U') \circ (V \star \phi') : V \circ V' \circ R'' \implies V \circ R' \circ U' \implies R \circ U \circ U'$, and its adjoint is $\Psi := (\psi \star V') \circ (U \star \psi')$.

Proof. It suffices to apply the definition 4.7.4.16 from [HA] to the simplicial set $S = \Delta^1$ to produce the required ∞-category. ☐

Proposition A.1.4. *Let $(\mathcal{C}_i, \mathcal{D}_i, L_i, R_i)_{i \in I}$ be a small diagram in the ∞-category $\mathcal{BC}$ (i.e. each $L_i \dashv R_i$ forms an adjunction between $\mathcal{C}_i$ and $\mathcal{D}_i$, and they are related by Beck-Chevalley squares indexed by the edges of I). Then this diagram admits a limit which is moreover computed component-wise on the objects. In particular, for each vertex $i \in I$, there is a canonical Beck-Chevalley square:*

$$\begin{tikzcd}
\lim_i \mathcal{C}_i \arrow[r, shift right, "\lim_i R_i"'] \arrow[r, phantom, "\perp"] \arrow[d, "U_i"'] & \lim_i \mathcal{D}_i \arrow[l, shift right, "\lim_i L_i"'] \arrow[d, "V_i"] \\
\mathcal{C}_i \arrow[r, shift right, "R_i"'] \arrow[r, phantom, "\perp"] & \mathcal{D}_i \arrow[l, shift right, "L_i"']
\end{tikzcd}$$

where U_i and V_i are the canonical projection maps from the respective limits of $\mathcal{C}_i$ and $\mathcal{D}_i$.

Proof. It suffices to apply the corollary 4.7.4.18 from [HA] to the simplicial set $S = \Delta^1$. ☐

Moreover, we will also need the following lemma which we will prove.

Lemma A.1.5. *In the context of reminder A.1.2, let ϕ be a right transformation satisfying the Beck-Chevalley condition, and let ψ be its adjoint. Then, the following diagram is commutative:*

$$\begin{tikzcd}
U \circ L' \circ R' \arrow[dr, "U \star \epsilon'"'] & L \circ V \circ R' \arrow[l, "\psi \star R'"', "\simeq"] \arrow[r, "L \star \phi", "\simeq"'] & L \circ R \circ U \arrow[dl, "\epsilon \star U"] \\
& U &
\end{tikzcd}$$

In other words, U sends the co-unit of $L' \dashv R'$ to the co-unit of $L \dashv R$ applied to U. Similarly, V sends the unit of $L' \dashv R'$ to the unit of $L \dashv R$ applied to V.

Proof. We will prove the statement for U since the proof for V is completely analogous. More precisely, it means that we will prove that $(U \star \epsilon') \circ (\psi \star R') = (\epsilon \star U) \circ (L \star \phi)$ as natural transformations $L \circ V \circ R' \implies U$. Since by assumption ψ is the left adjoint to ϕ, by definition we have $\psi := (\epsilon \star (U \circ L')) \circ (L \star \phi \star L') \circ ((L \circ V) \star \eta')$. In particular, using the associativity and distributivity laws of the vertical and horizontal compositions of natural transformations, and the triangle identity $(R' \star \epsilon') \circ (\eta' \star R') = Id_{R'}$, we can compute as required:

$$\begin{aligned}
&(U \star \epsilon') \circ (\psi \star R') \\
=\ &(U \star \epsilon') \circ (((\epsilon \star (U \circ L')) \circ (L \star \phi \star L') \circ ((L \circ V) \star \eta')) \star R') \\
=\ &(U \star \epsilon') \circ (\epsilon \star (U \circ L' \circ R')) \circ (L \star \phi \star (L' \circ R')) \circ ((L \circ V) \star \eta' \star R') \\
=\ &(\epsilon \star U) \circ (L \star \phi) \circ ((L \circ V \circ R') \star \epsilon') \circ ((L \circ V) \star \eta' \star R') \\
=\ &(\epsilon \star U) \circ (L \star \phi) \circ ((L \circ V) \star ((R' \star \epsilon') \circ (\eta' \star R'))) \\
=\ &(\epsilon \star U) \circ (L \star \phi) \circ ((L \circ V) \star Id_{R'}) \\
=\ &(\epsilon \star U) \circ (L \star \phi) \circ (Id_{L \circ V \circ R'}) \\
=\ &(\epsilon \star U) \circ (L \star \phi)
\end{aligned}$$

The reader familiar with string diagrams computations is invited to use them to follow along. ☐

A.2 Beck-Chevalley Squares in Derived Algebraic Geometry

In this section, we will look at instances of Beck-Chevalley conditions in the context of derived algebraic geometry. We start from a general adjunction between quasi-coherent algebras and relatively affine derived stacks, and arrive at two conditions, including one similar to the base-change formula (in definition A.2.4). We will also find sufficient conditions on maps for these Beck-Chevalley conditions to hold, which will apply in the context of section 5.2.

Reminder A.2.1. Let $f : V \to U$ be an arbitrary map between derived stacks, and consider the following diagram, as described in reminder 5.1.3 and reminder 5.1.4, in which a priori no commutativity holds, which we emphasize by inserting a question mark in its center:

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{QCoh}(V))^{\mathrm{op}} \arrow[d, shift right=2, "f_*"'] \arrow[d, phantom, "\dashv"] & & \mathrm{dSt}\,/V \arrow[ll, shift right=1.5, "\mathcal{O}_{/V}"'] \arrow[ll, phantom, "\perp"] \arrow[d, shift right=2, "f_!"'] \arrow[d, phantom, "\dashv"] \\
\mathrm{CAlg}(\mathrm{QCoh}(U))^{\mathrm{op}} \arrow[u, shift right=2, "f^*"'] \arrow[rr, shift right=1.5, "\mathrm{Spec}_U"'] & ? & \mathrm{dSt}\,/U \arrow[ll, shift right=1.5, "\mathcal{O}_{/U}"'] \arrow[ll, phantom, "\perp"] \arrow[u, shift right=2, "f^*"']
\end{tikzcd}$$

with $\mathrm{CAlg}(\mathrm{QCoh}(V))^{\mathrm{op}} \xrightarrow{\mathrm{Spec}_V} \mathrm{dSt}\,/V$ along the top.

As stated in reminder 5.1.4, in the above situation, the following diagrams are always commutative:

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{QCoh}(V))^{\mathrm{op}} \arrow[d, "f_*"'] & \mathrm{dSt}\,/V \arrow[l, "\mathcal{O}_{/V}"'] \arrow[d, "f_!"] \\
\mathrm{CAlg}(\mathrm{QCoh}(U))^{\mathrm{op}} & \mathrm{dSt}\,/U \arrow[l, "\mathcal{O}_{/U}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{CAlg}(\mathrm{QCoh}(V))^{\mathrm{op}} \arrow[r, "\mathrm{Spec}_V"'] & \mathrm{dSt}\,/V \\
\mathrm{CAlg}(\mathrm{QCoh}(U))^{\mathrm{op}} \arrow[u, "f^*"'] \arrow[r, "\mathrm{Spec}_U"'] & \mathrm{dSt}\,/U \arrow[u, "f^*"']
\end{tikzcd}$$

Remark A.2.2. In particular, we can wonder when the left square is right adjointable (resp. the right square is left adjointable), i.e. whether or not the two following squares commute:

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{QCoh}(V))^{\mathrm{op}} \arrow[r, "\mathrm{Spec}_V"'] \arrow[d, "f_*"'] \arrow[dr, phantom, "?"] & \mathrm{dSt}\,/V \arrow[d, "f_!"] \\
\mathrm{CAlg}(\mathrm{QCoh}(U))^{\mathrm{op}} \arrow[r, "\mathrm{Spec}_U"'] & \mathrm{dSt}\,/U
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{CAlg}(\mathrm{QCoh}(V))^{\mathrm{op}} \arrow[dr, phantom, "?"] & \mathrm{dSt}\,/V \arrow[l, "\mathcal{O}_{/V}"'] \\
\mathrm{CAlg}(\mathrm{QCoh}(U))^{\mathrm{op}} \arrow[u, "f^*"'] & \mathrm{dSt}\,/U \arrow[l, "\mathcal{O}_{/U}"'] \arrow[u, "f^*"']
\end{tikzcd}$$

We will therefore consider two definitions.

Definition A.2.3. *We say that a map of derived stacks $f : V \to U$* preserves relative spectra by push-forward *if the canonical natural transform $f_! \circ \mathrm{Spec}_V \to \mathrm{Spec}_U \circ f_*$, induced by reminder A.2.1, is an equivalence (i.e. if the left square of this remark is Beck-Chevalley). More explicitly, for any algebra $A \in \mathrm{CAlg}(\mathrm{QCoh}(V))$, there is always a canonical commutative diagram:*

$$\begin{tikzcd}
\mathrm{Spec}_V(A) \arrow[r, "\phi_A"] \arrow[d] & \mathrm{Spec}_U(f_*A) \arrow[d] \\
V \arrow[r, "f"'] & U
\end{tikzcd}$$

and we require that the top map ϕ_A is an equivalence for all A.

Definition A.2.4. *We say that a map of derived stacks $f : V \to U$* preserves relative structural sheaves by pull-back *if the canonical natural transform $f^* \circ \mathcal{O}_{/U} \to \mathcal{O}_{/V} \circ f^*$, induced by reminder A.2.1, is an equivalence (i.e. if the right square of this remark is Beck-Chevalley). More explicitly, for any $q : T \to U$, there is always a canonical pull-back diagram:*

$$\begin{tikzcd}
f^*T \arrow[r, "f'"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "q"] \\
V \arrow[r, "f"'] & U
\end{tikzcd}$$

and a canonical map $f^*q_*\mathcal{O}_T \to p_*f'^*\mathcal{O}_T$ of algebras in $\mathrm{QCoh}(V)$*, which we require that to be an equivalence for all T.*

We will now provide conditions on a map f that are sufficient to satisfy these properties. While the first condition can hold under reasonable conditions, the second one will require a refinement, as it is too strict when formulated as above. Let us thus start with the first one.

Lemma A.2.5. *If $f : V \to U$ is a quotient of a map between derived stacks relatively affine over* $\mathrm{Spec}(k)$[1] *by an affine group scheme, then it preserves relative spectra by push-forward (see definition A.2.3).*

Proof. Let us first only assume that $f : V \to U$ is a map between derived stacks relatively affine over $\mathrm{Spec}(k)$. We will later generalize it to the case of a quotient by an affine group scheme. We will now assume that $U = \mathrm{Spec}_k(B)$ and $V = \mathrm{Spec}_k(C)$ for some possibly non-connective k-cdga's B and C, that $f = \mathrm{Spec}_k(\phi)$ for some $\phi : B \to C$ a map of k-cdgas. In this context, let A be a C-algebra. We have to show that the canonical map $f_! \mathrm{Spec}_C(A) \to \mathrm{Spec}_B(f_* A)$ is an equivalence of derived stacks over $\mathrm{Spec}(B)$, i.e. that the top map of the following canonical commutative square is an equivalence:

$$\begin{tikzcd}
\mathrm{Spec}_C(A) \arrow[r, "\simeq?"] \arrow[d] & \mathrm{Spec}_B(f_* A) \arrow[d] \\
\mathrm{Spec}(C) \arrow[r] & \mathrm{Spec}(B)
\end{tikzcd}$$

In this case, we can give an explicit description of these stacks and this map. Let $\mathrm{Spec}(O)$ be an affine, by definition we have:

- $\mathbf{Map}_{\mathrm{dSt}}(\mathrm{Spec}(O), \mathrm{Spec}(B)) = \mathbf{Map}_{cdga}(B, O)$
- $\mathbf{Map}_{\mathrm{dSt}}(\mathrm{Spec}(O), \mathrm{Spec}(C)) = \mathbf{Map}_{cdga}(C, O)$
- Given a structural map $\mathrm{Spec}(O) \to \mathrm{Spec}(C) \in \mathrm{dSt} / \mathrm{Spec}(C)$, we have: $\mathbf{Map}_{\mathrm{dSt} / \mathrm{Spec}(C)}(\mathrm{Spec}(O), \mathrm{Spec}_C(A)) = \mathbf{Map}_{C-Alg}(A, O)$
- Given a structural map $\mathrm{Spec}(O) \to \mathrm{Spec}(B) \in \mathrm{dSt} / \mathrm{Spec}(B)$, we have: $\mathbf{Map}_{\mathrm{dSt} / \mathrm{Spec}(B)}(\mathrm{Spec}(O), \mathrm{Spec}_B(f_* A)) = \mathbf{Map}_{B-Alg}(f_* A, O)$

In the last two items, some data is in fact redundant. Let us look at $\mathrm{Spec}_C(A)$ for example, in which case the required data is two dashed arrows forming a commutative triangle (the plain arrow is fixed, it is the structural map $C \to A$):

$$\begin{tikzcd}
O & A \arrow[l, dashed] \\
 & C \arrow[u] \arrow[ul, dashed]
\end{tikzcd}$$

---

[1] As noted in remark 4.3.6, we mean here that U and V are affine stacks in the sense of [Toë06, §2.1], so we mean that $U = \mathrm{Spec}_k(B)$ and $V = \mathrm{Spec}_k(C)$ for B and C two possibly non-connective k-cdga's. Equivalently, we mean that the terminal maps $U, V \to \mathrm{Spec}(k)$ are relatively affine in the usual sense.

This data is equivalent to the data of a single map $A \to O$, since by commutativity the induced map $C \to O$ has to be the composition $C \to A \to O$. Moreover, in the case of $\mathrm{Spec}_B(f_*A)$, note that f_*A is equivalent to A as a cdga, as in this context the notation f_*A only serves to indicate that we equip it with the fixed structural map $B \to A$ given by the composition $B \to C \to A$. Therefore, in this context where everything is affine, $f_! \mathrm{Spec}_C(A)$ and $\mathrm{Spec}_B(f_*A)$ are equivalent to $\mathrm{Spec}(A)$ as derived stacks, and under this equivalence the canonical map $f_! \mathrm{Spec}_C(A) \to \mathrm{Spec}_B(f_*A)$ is the identity, i.e. the following square of derived stacks commutes:

$$\begin{tikzcd}
\mathrm{Spec}(A) \arrow[r, equal] & \mathrm{Spec}(A) \\
f_! \mathrm{Spec}_C(A) \arrow[r] \arrow[u, "\simeq"] & \mathrm{Spec}_B(f_*A) \arrow[u, "\simeq"']
\end{tikzcd}$$

This shows that the bottom map is also an equivalence as required.

Let us assume now that $f : V \to U$ is not a map of derived stacks relatively affine over $\mathrm{Spec}(k)$, but a quotient of such a map $g : V' \to U'$ by an affine group scheme G (i.e. $U = [U'/G]$, $V = [V'/G]$, $f = [g/G]$, and U', V' and G are derived stacks relatively affine over $\mathrm{Spec}(k)$). We are thus assuming that we have the following colimit diagram, where each of the two columns is a simplicial object:

$$\begin{tikzcd}
{[V'/G] = V} \arrow[r, "f"] & {U = [U'/G]} \\
V' \arrow[r, "g_0 = g"'] \arrow[u] & U' \arrow[u] \\
V' \times G \arrow[r, "g_1"'] \arrow[u] & U' \times G \arrow[u] \\
V' \times G^2 \arrow[r, "g_2"'] \arrow[u] & U' \times G^2 \arrow[u] \\
\dots \arrow[u] & \dots \arrow[u]
\end{tikzcd}$$

Recall from proposition A.1.3 that the ∞-category $\mathcal{BC}$ of adjunctions as objects and Beck-Chevalley squares as morphisms admits limits that are computed pointwise, so that its arrow ∞-category also has similar limits. For $n \in \mathbb{N}$, we will denote by V_n (resp. U_n) the following adjunctions, which are seen as objects of $\mathcal{BC}$:

$$V_n :$$

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{QCoh}(V' \times G^n))^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_{V' \times G^n}"'] & \mathrm{dSt}\,/V' \times G^n \arrow[l, shift right, "\mathcal{O}_{/V' \times G^n}"']
\end{tikzcd}$$

$$U_n :$$

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{QCoh}(U' \times G^n))^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_{U' \times G^n}"'] & \mathrm{dSt}\,/U' \times G^n \arrow[l, shift right, "\mathcal{O}_{/U' \times G^n}"']
\end{tikzcd}$$

Now at each level $n \in \mathbb{N}$, since G and the map g are relatively affine, g_n is relatively affine too, and the following square is Beck-Chevalley by the preceding case of this lemma:

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{QCoh}(V' \times G^n))^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_{V' \times G^n}"'] \arrow[d, "(g_n)_*"'] & \mathrm{dSt}\,/V' \times G^n \arrow[l, shift right, "\mathcal{O}_{/V' \times G^n}"'] \arrow[d, "(g_n)_!"] \\
\mathrm{CAlg}(\mathrm{QCoh}(U' \times G^n))^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_{U' \times G^n}"'] & \mathrm{dSt}\,/U' \times G^n \arrow[l, shift right, "\mathcal{O}_{/U' \times G^n}"']
\end{tikzcd}$$

We will call C_n this Beck-Chevalley square, thus $C_n : V_n \to U_n$ is an arrow in the ∞-category of $\mathcal{BC}$, or equivalently it is an object in the arrow ∞-category of $\mathcal{BC}$. Moreover, we will now describe transition maps from C_n to C_{n+1}. Note that the following square is Beck-Chevalley too by the same argument since both $V' \times G^{n+1}$ and $V' \times G^n$ are relatively affine over $\mathrm{Spec}(k)$ (the same is true in the analogous case for U'):

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{QCoh}(V' \times G^{n+1}))^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_{V' \times G^{n+1}}"'] \arrow[d] & \mathrm{dSt}\,/V' \times G^{n+1} \arrow[l, shift right, "\mathcal{O}_{/V' \times G^{n+1}}"'] \arrow[d] \\
\mathrm{CAlg}(\mathrm{QCoh}(V' \times G^n))^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_{V' \times G^n}"'] & \mathrm{dSt}\,/V' \times G^n \arrow[l, shift right, "\mathcal{O}_{/V' \times G^n}"']
\end{tikzcd}$$

This provides arrows $V_n \to V_{n+1}$ and $U_n \to U_{n+1}$, and moreover the following diagram commutes:

$$\begin{tikzcd}
V_n \arrow[r, "C_n"] \arrow[d] & U_n \arrow[d] \\
V_{n+1} \arrow[r, "C_{n+1}"'] & U_{n+1}
\end{tikzcd}$$

Therefore, we have the following co-simplicial object in the ∞-category of Beck-Chevalley squares (the arrow ∞-category of $\mathcal{BC}$):

$$\begin{tikzcd}
C_0 \arrow[d] \\
C_1 \arrow[d] \\
C_2 \arrow[d] \\
\vdots
\end{tikzcd}$$

By proposition A.1.4, it admits a limit C, computed pointwise, which is also a Beck-Chevalley square. Moreover, by definition and using the pointwise limit, we can immediately see that C is the following square:

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{QCoh}(V)^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_V"'] \arrow[d, "f_*"'] & \mathrm{dSt}\,/V \arrow[l, shift right, "\mathcal{O}_{/V}"'] \arrow[d, "f_!"] \\
\mathrm{CAlg}(\mathrm{QCoh}(U)^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_U"'] & \mathrm{dSt}\,/U \arrow[l, shift right, "\mathcal{O}_{/U}"']
\end{tikzcd}$$

The fact that this square is Beck-Chevalley concludes this lemma. ☐

As explained earlier, we will now turn to the second condition, but we will have to refine it.

Remark A.2.6. Let $f : V \to U$ be a map of derived stacks, and suppose we want to prove that it preserves relative structural sheaves by pull-back (see definition A.2.4). Let $q : T \to U$ be any map of derived stacks, we can form the following canonical pull-back square:

$$\begin{tikzcd}
f^*T \arrow[r, "f'"] \arrow[d, "q'"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "q"] \\
V \arrow[r, "f"'] & U
\end{tikzcd}$$

We have to show that the following canonical map of commutative algebras in QCoh(V) is an equivalence:

$$f^*p'_*\mathcal{O}_T \to p_*f'^*\mathcal{O}_T$$

This statement is a base-change formula, and we are requiring it to be true along any arbitrary map $q : T \to U$, which need not be true under reasonable assumptions in this level of generality. However, as is well-known in algebraic geometry, we will show that it is true when the statement is restricted to maps $q : T \to U$ that are relatively affine, and $f : V \to U$ is finite.

Definition A.2.7. *Given a derived stack S, the essential image of* Spec_S *will be denoted by:*

$$\mathrm{RelAff}/S$$

Remark A.2.8. As in reminder A.2.1, given $f : U \to V$ any map of derived stacks, we can construct the following square:

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{QCoh}(U))^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_U"'] \arrow[d, "f^*"] & \mathrm{dSt}\,/U \arrow[l, shift right, "\mathcal{O}_{/U}"'] \arrow[l, phantom, "\perp"] \arrow[d, "f^*"'] \\
\mathrm{CAlg}(\mathrm{QCoh}(V))^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_V"'] & \mathrm{dSt}\,/V \arrow[l, shift right, "\mathcal{O}_{/V}"'] \arrow[l, phantom, "\perp"]
\end{tikzcd}$$

Moreover, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{QCoh}(U))^{\mathrm{op}} \arrow[r, "\mathrm{Spec}_U"'] \arrow[d, "f^*"] & \mathrm{dSt}\,/U \arrow[d, "f^*"'] \\
\mathrm{CAlg}(\mathrm{QCoh}(V))^{\mathrm{op}} \arrow[r, "\mathrm{Spec}_V"'] & \mathrm{dSt}\,/V
\end{tikzcd}$$

As a consequence, given $q : T \to U$, if T lies in RelAff/U, i.e. if there exists $A \in \mathrm{CAlg}(\mathrm{QCoh}(U))$ such that $T \simeq \mathrm{Spec}_U(A)$ as derived stacks over U, then $f^*T \simeq \mathrm{Spec}_V(f^*A)$ as derived stacks over V, i.e. f^*T lies in RelAff/V. In conclusion, the previous square restricts to the following one:

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{QCoh}(U))^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_U"'] \arrow[d, "f^*"] & \mathrm{RelAff}/U \arrow[l, shift right, "\mathcal{O}_{/U}"'] \arrow[l, phantom, "\perp"] \arrow[d, "f^*"'] \\
\mathrm{CAlg}(\mathrm{QCoh}(V))^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_V"'] & \mathrm{RelAff}/V \arrow[l, shift right, "\mathcal{O}_{/V}"'] \arrow[l, phantom, "\perp"]
\end{tikzcd}$$

Definition A.2.9. *We will say that a map of derived stacks $f : V \to U$ preserves the relative structural sheaves of relative affines by push-forward if the above square is Beck-Chevalley, i.e. if when restricted to* RelAff/U*, the canonical natural transform:*

$$f^* \circ \mathcal{O}_U \to \mathcal{O}_V \circ f^*$$

is an equivalence.

We will now give a reasonable condition on f for this weaker condition to hold.

Lemma A.2.10. *If $f : V \to U$ is a finite map of derived stacks, it preserves the relative structural sheaves of relative affines by push-forward.*

Proof. Let $q : T \to U$ be a relatively affine map of derived stacks, we can form the following canonical pull-back square:

$$\begin{tikzcd}
f^*T \arrow[r, "f'"] \arrow[d, "q'"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "q"] \\
V \arrow[r, "f"'] & U
\end{tikzcd}$$

We have to show that the following canonical map of commutative algebras in $\mathrm{QCoh}(V)$ is an equivalence:

$$f^* p'_* \mathcal{O}_T \to p_* f'^* \mathcal{O}_T$$

However, since q is relatively affine, it is quasi-compact and quasi-separated, and by the assumption that $f : V \to U$ is finite, we can apply [HP23, Lem. A.1.3], which yields the result. □

Remark A.2.11. While the first condition provided a Beck-Chevalley square with the ∞-categories $\mathrm{dSt}\,/V$ and $\mathrm{dSt}\,/U$ appearing on the right-hand side (see definition A.2.4), the previous condition provides a Beck-Chevalley square where the right-hand side is restricted to RelAff/V and RelAff/U (see definition A.2.9). As such, these squares cannot be composed as-is. However, we can see that the first condition restricts to relative affines. Indeed, let $f : V \to U$ be a map of derived stacks which preserves relative spectra by push-forward. This means that the following square is Beck-Chevalley:

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{QCoh}(V))^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_V"'] \arrow[r, phantom, "\perp"] \arrow[d, "f_*"'] & \mathrm{dSt}\,/V \arrow[l, shift right, "\mathcal{O}_{/V}"'] \arrow[d, "f_!"] \\
\mathrm{CAlg}(\mathrm{QCoh}(U))^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_U"'] \arrow[r, phantom, "\perp"] & \mathrm{dSt}\,/U \arrow[l, shift right, "\mathcal{O}_{/U}"']
\end{tikzcd}$$

In particular, for any $A \in \mathrm{CAlg}(\mathrm{QCoh}(V))$, we have that $f_! \operatorname{Spec}_V(A) \simeq \operatorname{Spec}_U(f_* A)$ as derived stacks over U. As a consequence, it restricts to the following Beck-Chevalley square:

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{QCoh}(V))^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_V"'] \arrow[r, phantom, "\perp"] \arrow[d, "f_*"'] & \mathrm{RelAff}/V \arrow[l, shift right, "\mathcal{O}_{/V}"'] \arrow[d, "f_!"] \\
\mathrm{CAlg}(\mathrm{QCoh}(U))^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_U"'] \arrow[r, phantom, "\perp"] & \mathrm{RelAff}/U \arrow[l, shift right, "\mathcal{O}_{/U}"']
\end{tikzcd}$$

In conclusion of this appendix, we arrived at the following property:

Conclusion A.2.12. Let $f : V \to U$ be a map of derived stacks.

1. By lemma A.2.5, if f is a quotient of a map between derived stacks relatively affine over $\mathrm{Spec}(k)$ by an affine group scheme, then the following square is Beck-Chevalley:

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{QCoh}(V))^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_V"'] \arrow[r, phantom, "\perp"] \arrow[d, "f_*"'] & \mathrm{RelAff}/V \arrow[l, shift right, "\mathcal{O}_{/V}"'] \arrow[d, "f_!"] \\
\mathrm{CAlg}(\mathrm{QCoh}(U))^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_U"'] \arrow[r, phantom, "\perp"] & \mathrm{RelAff}/U \arrow[l, shift right, "\mathcal{O}_{/U}"']
\end{tikzcd}$$

2. By lemma A.2.10, if f is finite, then the following square is Beck-Chevalley:

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{QCoh}(U))^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_U"'] \arrow[r, phantom, "\perp"] \arrow[d, "f^*"] & \mathrm{RelAff}/U \arrow[l, shift right, "\mathcal{O}_{/U}"'] \arrow[d, "f^*"'] \\
\mathrm{CAlg}(\mathrm{QCoh}(V))^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}_V"'] \arrow[r, phantom, "\perp"] & \mathrm{RelAff}/V \arrow[l, shift right, "\mathcal{O}_{/V}"']
\end{tikzcd}$$

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Titre : Structures symplectiques avec décalage adaptées à un feuilletage

Mots clés : Géométrie algébrique dérivée, Théorie des feuilletages, Géométrie symplectique

Résumé : Dans cette thèse, nous travaillons en géométrie algébrique dérivée en caractéristique nulle et étudions les feuilletages dérivés ainsi que les structures symplectiques avec décalage sur de tels objets. Notre premier résultat principal est un théorème d'existence pour les feuilletages dérivés sur les champs de morphismes. Plus précisément, si S est un champ dérivé de base, X est un champ dérivé sur S schématique-propre, plat et localement d'intersection complète, et Y est un champ dérivé de Deligne-Mumford relatif sur S muni d'un feuilletage dérivé, alors le champ dérivé des morphismes Map_S(X,Y) hérite lui-même, sous des hypothèses naturelles de représentabilité, d'un feuilletage dérivé. Afin d'établir ce résultat, nous montrons d'abord que l'infini-catégorie des feuilletages dérivés relatifs peut être réalisée comme une sous-catégorie pleine d'une infini-catégorie de champs dérivés munis d'une structure supplémentaire, puis nous démontrons que la restriction de Weil le long d'un morphisme schématique-propre, plat et localement d'intersection complète préserve cette structure. On en déduit, plus généralement, un foncteur d'image directe pour les feuilletages dérivés, ainsi qu'une description explicite de leurs complexes tangents après image directe. Comme applications, nous obtenons des feuilletages dérivés sur certains espaces de modules dérivés, notamment les espaces de modules dérivés de courbes stables et les schémas de Hilbert dérivés.

La seconde partie de la thèse introduit les nouvelles notions de structures symplectiques décalées sur les feuilletages dérivés et de feuilletages dérivés lagrangiens. Nous démontrons ensuite des théorèmes d'existence pour ces structures : un théorème d'intersection pour les feuilletages dérivés lagrangiens dans un champ dérivé symplectique n-décalé, produisant un feuilletage dérivé symplectique (n-1)-décalé, ainsi qu'un théorème d'image directe symplectique, montrant que l'image directe d'un feuilletage dérivé symplectique n-décalé le long d'un morphisme d-orienté porte naturellement une structure symplectique (n-d)-décalée.

Title: Shifted Symplectic Structures on Derived Foliations

Key words: Derived Algebraic Geometry, Foliation Theory, Symplectic Geometry

Abstract: In this thesis, we work in derived algebraic geometry in characteristic zero and study derived foliations and shifted symplectic structures on such objects. Our first main result is an existence theorem for derived foliations on mapping stacks. More precisely, if S is a base derived stack, X is a proper-schematic, flat, and local complete intersection derived stack over S, and Y is a relative derived Deligne-Mumford stack over S equipped with a derived foliation, then the derived mapping stack Map_S(X,Y) itself inherits, under natural representability hypotheses, a derived foliation. In order to establish this result, we first show that the infinity-category of relative derived foliations can be realized as a full subcategory of an infinity-category of derived stacks equipped with extra structure, and then prove that the Weil restriction along a proper-schematic, flat, and local complete intersection morphism preserves this structure. This yields, more generally, a push-forward functor for derived foliations, together with an explicit description of their tangent complexes after push-forward. As applications, we obtain derived foliations on certain derived moduli spaces, including derived moduli spaces of stable curves and derived Hilbert schemes.

The second part of the thesis introduces the new notions of shifted symplectic structures on derived foliations and Lagrangian derived foliations. We then prove existence theorems for these structures: an intersection theorem for Lagrangian derived foliations in an n-shifted symplectic derived stack, producing an (n-1)-shifted symplectic derived foliation, and a symplectic push-forward theorem, showing that the push-forward of an n-shifted symplectic derived foliation along a d-oriented morphism naturally carries an (n-d)-shifted symplectic structure.